Mid-term Quiz

Game Theoretical Methodology and
Technique for Internet Protocols

October 24, 2016

Problem 1

Now, we have two players in this game, player M and player N, and each of them have two strategies A,B. Below is an payoff bi-matrix, the row player M, and the column player N.

	A	B
A	2,3	5,2
B	3,1	1,5

Table 1: Payoff matrix for player M & N

a) Please find the Nash Equilibrium with fixed point theorem of the payoff bi-matrix above, and the expected revenue.

b) Please talk about the relationship of Nash Equilibrium and fixed point.

Problem 2

Consider a single-item auction with n bidders. Their values of the item follow (not necessarily identical) independent distributions denoted by $F_1, \ldots, F_n$. Recall the Myerson's auction which is elegantly implementable when we merely consider i.i.d condition while it is not quite feasible when we relax this assumption.

a) Write down the form of the expected optimal revenue OPT with virtual valuation $\pi_i(t_i) = t_i - \frac{1-F_(t_i)}{f_i(t_i)}$

b) Design a strategy by modifying the Myerson's auction to obtain at least $C \cdot OPT$ expected revenue ($C \in (0, 1]$).

Problem 3

a) There are two boxes, each of which contains a positive sum of money. One box contains twice as much as the other. You may pick one box and keep whatever amount it contains. You pick one box at random but before you open it you are given the chance to take the other box instead. Now suppose you reason as follows:

1. I denote by A the amount in my selected box.
2. The probability that A is the smaller amount is 1/2, and that it is the larger amount is also 1/2.
3. The other box may contain either 2A or A/2.
4. If A is the smaller amount, then the other box contains 2A.
5. If A is the larger amount, then the other box contains A/2.
6. Thus the other box contains 2A with probability 1/2 and A/2 with probability 1/2.
7. So the expected value of the money in the other envelope is: $\frac{1}{2}(2A) + \frac{1}{2}(A/2) = \frac{5}{4}A$
8. This is greater than A, so I gain on average by swapping.
9. After the switch, I can denote that content by B and reason in exactly the same manner as above.
10. I will conclude that the most rational thing to do is to swap back again.
11. To be rational, I will thus end up swapping boxes indefinitely.

As it seems more rational to open just any box than to swap indefinitely, we have a contradiction. What has gone wrong? If you are the player, Will you swap or just open the box you choose? Please explain why.

b) With a little change, this time two rational players play boxes game as in (a). Each player gets a box, which contains a positive sum of money. One player's box has twice as much money as the other's but the players are not sure which one contains more. The money in both boxes has the same lower bound and upper bound (suppose money in box i is m_i, then $A \leq m_i \leq B$, where A and B are both constants) which are public to both players. Each player can open his box and know how much money is inside and then choose to exchange the box with the other player or not. They will give decisions independently and simultaneously. If both players agree to exchange, then the boxes are exchanged and each of them get money inside the other's box.

Problem 4

A zero-sum game of Blue side vs Red side is $G = \{S_1, S_2; A\}$ with Blue side's strategy set $S_1 = \{B_1, B_2\}$, Red side's strategy set $S_2 = \{R_1, R_2\}$, tradeoff matrix $A = \begin{pmatrix} 3 & 7 \\ 5 & 1 \end{pmatrix}$. Is there a Pure Nash Equilibrium for the game ? If so, please show it, Or tell why and show a Mixed Nash Equilibrium.

Problem 5

Using Brouwer Fixed Point Theorem to prove that Nash equilibrium in mixed strategies exist.

Problem 6

a) Prove the first-price auction is truthful or not.

b) Consider a specific buyer **B**, who has a private value v for the item. Denote the maximum bid price from other buyers as $z \in [0, \infty)$. Suppose z follows the distribution $p_z(z) = \frac{l}{(l+z)^2}$ where $l > 0$. What is the **B**'s optimal bid price for the item under the first-price auction?

Problem 7

1. Consider two sets of men and women with 4 elements each, their preferences of the opposite sex are listed as follows. Without loss of generality, men are arbitrarily and separately labeled A, B, C, D while women are labeled E, F, G, H. Find a stable matching in this case.

Men's Preferences						Women's Preferences				
A	E	F	G	H		E	B	D	A	C
B	G	F	E	H		F	A	B	C	D
C	G	H	E	F		G	D	C	A	B
D	G	F	H	E		H	D	B	A	C

2. If the size of both sets is n, then what is the time complexity to find a stable matching using the famous Gale-Shapley Algorithm (Men Proposal Algorithm)?

3. Prove that for every collection of preference lists for the men and women, there exists at least one stable matching.

4. Is Gale-Shapley Algorithm dominant-strategy incentive-compatible (DSIC) ?

Problem 8

Consider the following game between the four players A, B, C and D. Each has a choice two strategies I, II. The payment bi-matrix:

$$\text{A and B: } \begin{bmatrix} & I & II \\ I & 1,0 & 2,1 \\ II & -1,-1 & 0,2 \end{bmatrix}, \text{ A and C: } \begin{bmatrix} & I & II \\ I & 1,0 & -1,-1 \\ II & -1,0 & 1,-1 \end{bmatrix}, \text{ A and D: } \begin{bmatrix} & I & II \\ I & 0,-1 & 1,0 \\ II & -1,-1 & -1,1 \end{bmatrix}$$

$$\text{B and C: } \begin{bmatrix} & I & II \\ I & 0,3 & 1,2 \\ II & 2,1 & 3,0 \end{bmatrix}, \text{ B and D: } \begin{bmatrix} & I & II \\ I & 0,0 & 1,0 \\ II & 0,1 & 3,2 \end{bmatrix}, \text{ C and D: } \begin{bmatrix} & I & II \\ I & 1,0 & 2,1 \\ II & -1,-1 & 0,2 \end{bmatrix},$$

a) Find all Nash Equilibriums of player A and B, explain.

b) Find all Nash Equilibriums of players A, B, C and D, explain.

Problem 9

Consider a naive two-player zero-sum matrix game. There are two players, Alice and Bob. Each player has n pure strategies and Alice's gain is C_{ij} if Alice chooses her i-th strategy while Bob chooses his j-th strategy.

If C is non-negative and

$$\forall i, C_{ii} > n \times \sum_{j \neq i} C_{ij} \wedge C_{ii} > n \times \sum_{j \neq i} C_{ji}$$

, please **prove** that $a^T = \frac{e^T A^{-1}}{e^T A^{-1} e}$ and $b = \frac{A^{-1} e}{e^T A^{-1} e}$ is a Nash Equilibrium, where $e = (1, 1, \ldots, 1)^T$.

Please notice that

$$\forall i, C_{ii} > \sum_{j \neq i} C_{ij} \wedge C_{ii} > \sum_{j \neq i} C_{ji}$$

means a nonnegative matrix is strictly diagonally dominant, which implies this matrix is invertible.

Problem 10

Two companies are competing for the same product. To improve its market share, company A decides to launch the following strategies:

- A_1 = give discount coupons
- A_2 = home delivery
- A_3 = fee gifts

The company B decides to use media advertisement to promote its product with following strategies:

- B_1 = Internet
- B_2 = newspaper
- B_3 = magazine

	B_1	B_2	B_3
A_1	3	-4	2
A_2	1	-7	-3
A_3	-2	4	7

Use linear programming to determine the best strategies for both the companies.

Problem 11

100 rational bidders bid for one item in a Vickrey auction. The item's private value for bidder i is v_i, satisfying uniform distribution on $[0, 100]$. How much is the expectation of each bidder's payment?

Problem 12

Find the reduced form and CRF function and discuss allocation rules and pricing rules for the 2 independent bidders, one uniform in [0,5], another in [1,6].

Problem 13

	Swerve	Don't Swerve
Swerve	0,0	-1,1
Don't Swerve	T,-1	-2,-2

Table 2: Comparison Between Routing Algorithms

a) Find all of the pure strategy Nash equilibrium strategy profiles for this game if $T > 0$.

b) Find all of the pure strategy Nash equilibrium profiles for this game if $T < 0$.

c) If $T > 0$, there is a mixed strategy Nash equilibrium strategy profile that is not a pure strategy Nash equilibrium. Find it and find the payoffs to each player in this equilibrium.

d) In a mixed strategy Nash equilibrium with $T = 2$, which player is more likely to swerve? If $T = 2$, which player gets the higher expected payoff in equilibrium? Which players equilibrium mixed strategy depends on T.

Problem 14

a) Suppose a VCG mechanism is applied to sell the objects in $O = \{a, b\}$ to three buyers. A buyer can buy none, one, or both of the objects. For simplicity, assume the valuation function of each buyer depends only on the set of objects assigned to that buyer. The values are:

$$u1(\varnothing) = 0, u1(\{a\}) = 10, u1(\{b\}) = 3, u1(\{a, b\}) = 13$$

$$u2(\varnothing) = 0, u2(\{a\}) = 2, u2(\{b\}) = 8, u2(\{a, b\}) = 10$$

$$u3(\varnothing) = 0, u3(\{a\}) = 3, u3(\{b\}) = 2, u3(\{a, b\}) = 14$$

To be definite, suppose the standard version of the payment rule is used whereby for each buyer i, $m_i(u)$ is the maximum welfare of the other buyers minus the realized welfare of the other buyers, both computed using the reported valuation functions.Determine the assignment of objects to buyers and the payments of the buyers, under truthful bidding.And Discuss why buyer 3 might have an objection to the outcome.

b) Given a pdf f_i with support equal to the interval $[0,u_i](u_i > 0)$, the virtual valuation function is defined on the same interval and is given by $\psi_i(x_i) = x_i - \frac{1-F_i(x_i)}{f_i(x_i)}$.Show that $E[\psi_i(X_i)] = 0$.

c) Find the reduced form and CRF function and discuss allocation rules and pricing rules for two buyers in standard normal distribution.

Problem 15

There 5 players who want to allocate 100 gold coins. They decide to perform the allocation follow the steps below:

- **Step 1:** All the players line up to a queue and never change the order of the queue.
- **Step 2:** The first player in the queue propose an allocation and all the players in the queue vote for this allocation.
- **Step 3:** If ***more than*** half of the players vote yes, follow the allocation of the first player and the game ends. Otherwise, get rid of the first player from the queue and he can not get involved in the allocation any more, then go to the 2 step.

Suppose all the players are very smart and greedy, Questions:

a) If you are the first player of the 5 players, how will you propose the allocation? And why?

b) If there are 9 players and you are the first player, how will you propose the allocation? And why?

c) If there are n players and m coins $(m > n)$, how much will the first player get?