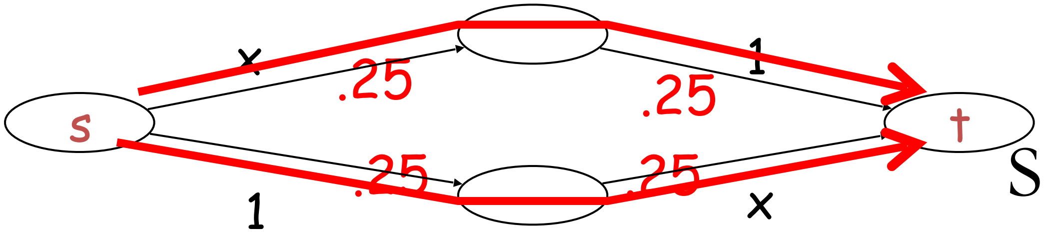
Lecture 10 Network Congestions

Xiaotie Deng
AIMS Lab, SJTU

Braess' Paradox

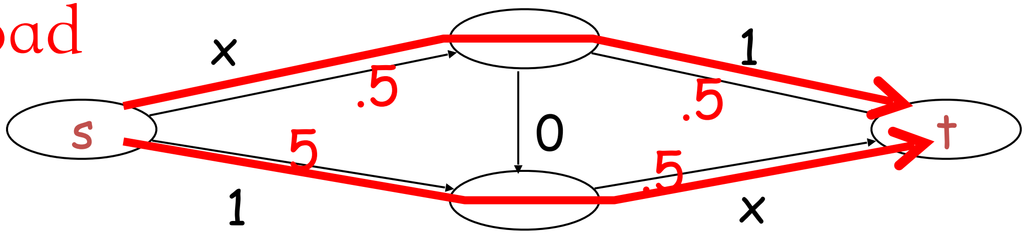
A Simple Road System：1／2

Original Network



Social Consumption＝
0.5＋0.125=0.625

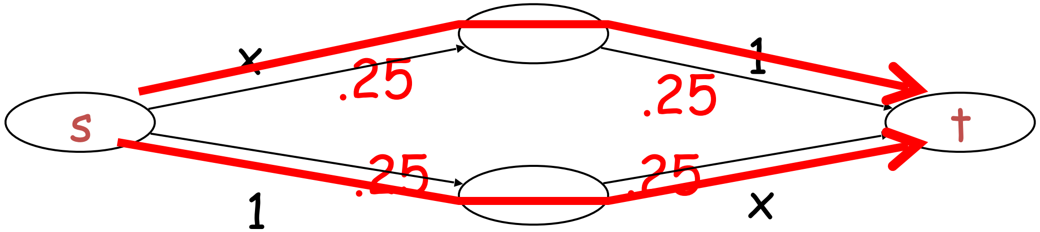
Add a fast road



What happens?

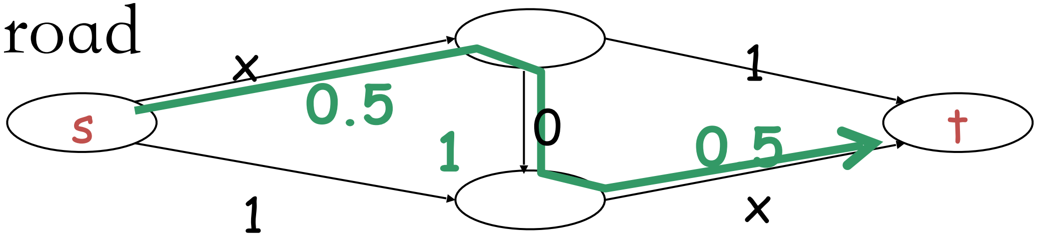
Add a speed road

Original Network



Total cost= 0.5+ 0.125=0.625

Add a speed road

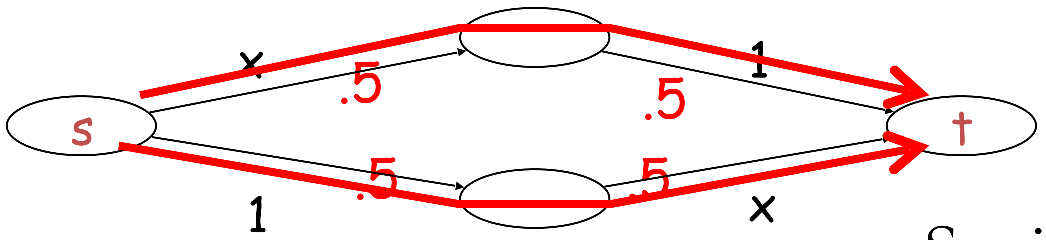


Consumption at Nash equilibrium=0.5

All runs fast and minimize total cost.

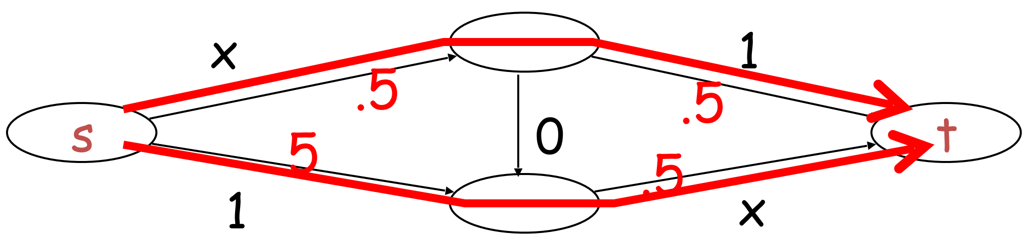
Bigger Traffic: 1

Original Network



Social Cost= 1.5

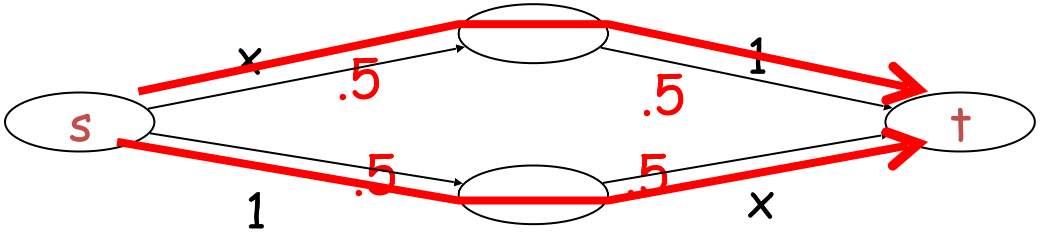
Add a Road



Would it improve?

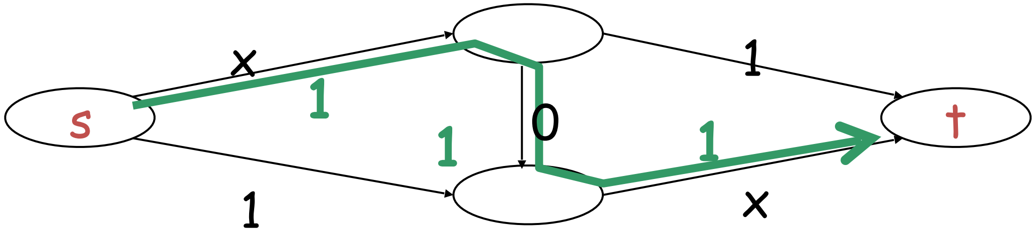
Braess' Paradox

Original roads



Total social cost= 1.5

Add a road



Total cost at equilibrium = 2

Traffic jam is worsen!

Resolution of Traffic Dilemma

- (Pigouvian Tax)
  - f*l(f):traffic times the first derivative of delay function。
- Result:
  - Nash equilibrium=Social optimum
- Effect
  - total delay minimized
  - no investment but improved revenue
  - individual optimum aligned with social optimum

Outline

- Inefficiency in Routing Games
- Potential Games
- Marginal Cost Pricing

Issues Involved with Equilibrium

- Social Optimal = outcome maximize the social welfare
  - Social Welfare: total utility of all players (the sum of individual utilities).
- Nash = outcome of individual rational (selfish) behavior
  - Nash may be inefficient in terms of social welfare.

Social Optimum

- Nash equilibrium

(4,4)

- Social optimum

(2,2)

P1 \ P2	Confess	Silent
Confess	4 (P1), 4 (P2)	1 (P1), 5 (P2)
Silent	5 (P1), 1 (P2)	2 (P1), 2 (P2)

Two Measures of Inefficiency with respect to Social Welfare

How far away an equilibrium ‘s social welfare can be from that of the social optimum?

$$\text{Price of Anarchy} = \frac{\text{cost of worst selfish outcome}}{\text{“socially optimum” cost}}$$

How close the best one can be from Social OPT?

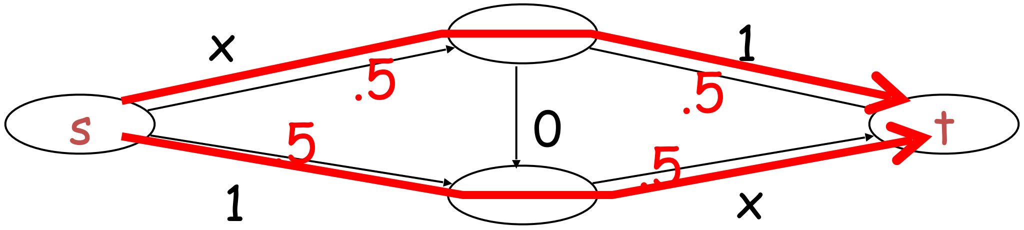
$$\text{Price of Stability} = \frac{\text{cost of best selfish outcome}}{\text{“socially optimum” cost}}$$

Flow in a Directed Network

- A directed graph with distinguished nodes s, t.
- Define: Flow
  - A collection of paths from s to t, possibly with a weight $\mathbf{f_p}$ on each path p: $f=\boldsymbol{\Sigma_{\text{all } p}\, f_p * 1_p}$
    - Where $\mathbf{1_p}$ is a characterization vector: $1^E \text{->} R^E$ being one on edges of the path p and zero on other edges
    - Representation may not be unique
    - Equivalent to require inflow=outflow on every node other than {s, t}.
  - Flow on an edge: $\boldsymbol{f_e=\Sigma_{\text{all } p:\, e \varepsilon P}\, f_P}$
  - Value of a flow: $\boldsymbol{\Sigma_{\text{all } p}\, f_p}$

Notations and Concepts

- Cost function on an edge
  - a function $\ell_e(x)$ of the load on the edge e
    - Flow of size ε on e incurs a cost $\varepsilon * \ell_e(x)$
  - usually assumed continuous, nonnegative, and monotone increasing in load $x=f_e$ on edge e
  - Example: cost a constant 1 or the flow size such as the congestion in a transportation network.



Routing Games

- Non-atomic routing games
  - Each user controls a negligible fraction of the flow.
  - A pure strategy of a user is a path
    - Its cost does not change when a player changes strategy
    - The cost of strategy p: $\ell_P(\mathbf{f}) = \Sigma_{e \varepsilon P}\, \ell_e(\mathbf{f}_e)$
- Atomic routing games
  - Unit case: Each user controls a unit of flow, and selects a single path
  - Weighted case: A user may control r units from s to t and routes it on a single path.
    - Its cost changes according to the flow after routing is done

Individual OPT v.s. Social OPT

Non-atomic flow:

Individual objective: Find a path P that minimizes the total delay

$$\text{Min } \ell_P(\mathbf{f}) = \Sigma_{e \varepsilon P}\, \ell_e(\mathbf{f}_e)$$

Social welfare:

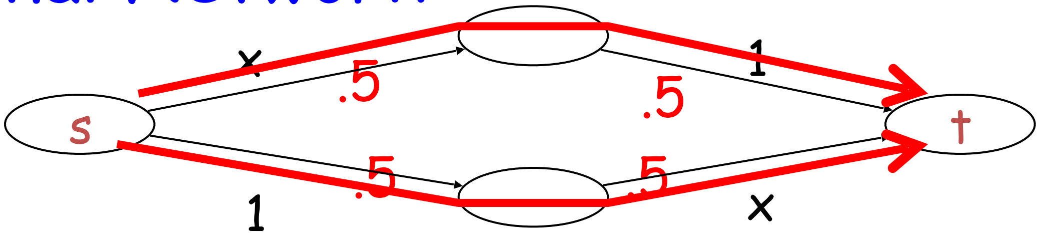
SW(f) = total latency of a flow **f**: $= \Sigma_P\, \mathbf{f}_P \bullet \ell_P(\mathbf{f})$

$$= \Sigma_e \mathbf{f}_e \bullet \ell_e(\mathbf{f}_e)$$

(since $\Sigma_e \Sigma_{P:\, e \varepsilon P}\ \mathbf{f}_P \bullet \ell_e(\mathbf{f}_e) = \Sigma_e \mathbf{f}_e \bullet \ell_e(\mathbf{f}_e)$)

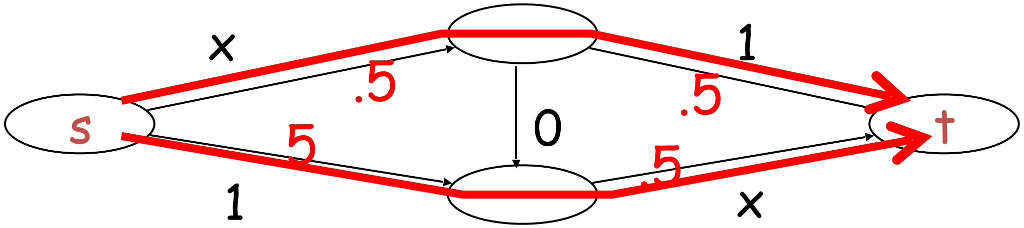
Braess's Paradox

Original Network



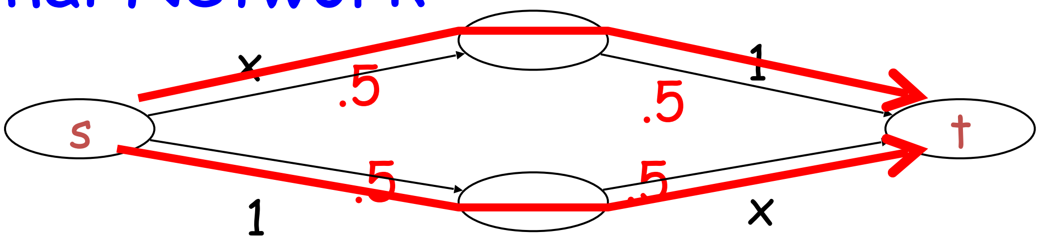
Cost of Nash flow = 1.5

Effect of building a new road ?



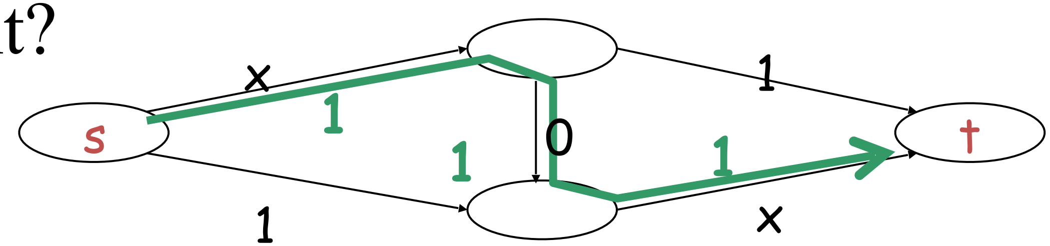
Braess's Paradox

Original Network



Cost of Nash flow = 1.5

Improvement?



Cost of Nash flow = 2

All the flow has increased delay!

What's the optimal solution?

Connecting Nash and Opt

- (OPT) minimize $C(f) = \Sigma_e f_e \bullet \ell_e(f_e)$
- subject to: f is an s-t flow
- carrying r units
  - Social optimum, call it "Optimal Flow"

- (Nash) Min-latency flow
  - Every s-t path P with $f_e>0$ for every e in P, is a shortest path in terms of the latency.

Characterizing the Optimal Flow

- Optimality condition: all flow travels along minimum-gradient paths (change edge weight to gradient and find shortest path from s to t)
- Proof: otherwise, there is a flow of some negligible amount from s to t can be switched from the longer path to a shorter path (both measured in gradients) so as to reduce the total cost.

gradient is:

$$g(x) = (x\,\ell(x))'$$
$$= \ell(x) + x\,\ell'(x)$$

Compare: a flow f is at Nash equilibrium iff all flow travels along minimum-latency paths

Computational equivalence: Nash ↔ Min-Cost

Corolary 1: min cost is "Nash" with delay

$\ell(x)+x\,\ell'(x)$ on an edge of flow size x.

Corollary 2: Nash is ''min cost'' with delay

$\int_0^{f_e} \ell_e(x)\,dx$ on an edge of flow size f_e.

$$\Phi(f) = \Sigma_e \int_0^{f_e} \ell_e(x)\,dx$$

Why?

(its gradient:)

$$\left(\int_0^{f_e} \ell_e(x)\,dx\right)' = \ell(f_e)$$

Potential function Φ

- Potential Function is a function on the states of the network such that
  - Removing a player reduces the cost
  - The potential function reduces by the same amount.

Potential function Φ

- Recall Nash is the optimal flow minimizing Φ

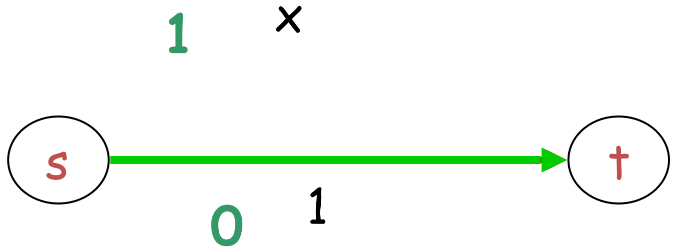
Theorem (Beckmann'56)

- In a network latency functions $\ell_e(x)$ that are monotone increasing and continuous,
  - A pure Nash equilibrium exists, and is essentially unique (i.e., if f and f' are equilibrium flows, then $\ell_e(f_e)=\ell_e(f_e')$)

Sharper results for non-atomic games

Theorem 1 (Roughghgarden-Tardos'00)

- In a network with linear latency functions
  - i.e., of the form $\ell_e(x)=a_e x+b_e$
- the cost of a Nash flow is at most 4/3 times that of the minimum-latency flow
- Lower bound: For n users of 1/n traffic each, everyone will choose the upper route to end up with a Nash equilibrium with unit cost 1 for all traffic.
- The optimum, on the other hand, route half to the upper route and another half to the lower half. The total cost will 0.5*(1+0.5)=3/4.
- Price of anarchy is at least 4/3.



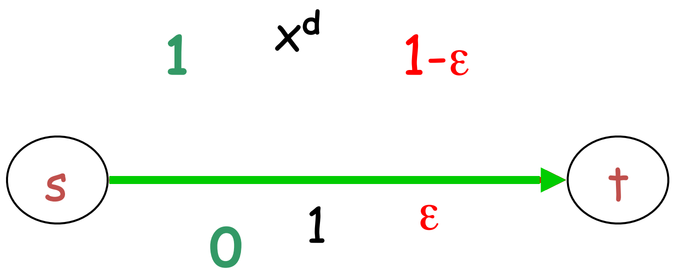
Sharper results for non-atomic games

Theorem 1 (Roughgarden-Tardos'00)

- In a network with linear latency functions
  - i.e., of the form $\ell_e(x)=a_e x+b_e$
- the cost of a Nash flow is at most 4/3 times that of the minimum-latency flow
- Proof: Left for after-class readings.

General Latency Functions

- Question: what about more general edge latency functions?
- Bad Example: (r = 1, d large)



A Nash flow can cost arbitrarily more than the optimal (min-cost) flow

How to reduce the price of anarchy?

- Can we design or modify a selfish routing network to minimize the inefficiency?
  - Add toll fee
  - Common in practice

Marginal Cost Pricing

- Add marginal cost taxes on all edges
- Idea: charge each network user on each edge for the additional cost its presence causes for the other users on that edge.
  - $l_e^t(x)=l_e(x)+t_e$
  - $t_e=f_e\, l_e'(f_e)$

  Theorem: Let $t_e=f_e^*\, l_e'(f_e^*)$, where f* is an optimal flow for the original game. Then f* is a Nash equilibrium flow for the original game with modified edge cost $l_e^t(x)$.