SIngle Item Double Auction

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Deterministic Double Auction (Vickrey)

One Item Exchange Market

- There are 1 buyer and 1 seller.
  - The value of the buyer is $v_1 \in [\alpha_1, \beta_1]$.
  - The value of the seller is $v_2 \in [\alpha_2, \beta_2]$.
- Let $o(b_1, b_2) = 1$ if the item is sold to the buyer where b_1 is the bid of buyer and b_2 is the bid of seller.
- Let $x(b_1, b_2) = (x_1, x_2)$ denote the price x_1 the buyer pays and the price x_2 the seller gets.

Desired Properties of Auction

- Ex-Post Efficiency: buyer wins ($o(b_1, b_2) = 1$) if and only if $b_1 > b_2$). It implies that $p(0^-, 0) = 0$ and $p(0^+, 0) = 1$
  - We use c^+ represent the upper limit of c

$$\lim_{\epsilon > 0, \epsilon \to 0} c + \epsilon$$

- Budget balancedness: buyer pays x_1 and seller receives x_2 satisfies $x_1(b_1, b_2) - x_2(b_1, b_2) \geq 0$.
- Truthful: It is optimal for both buyer and seller to report the truth.

Bid Independent Pricing

Property $x_1(b_1, b_2) = x_1(b_2^+, b_2)$ for $b_1 > b_2$
$x_2(b_1, b_2) = x_2(b_1, b_1^-)$ for $b_2 < b_1$.

- Let $a = x_1(0^+, 0)$, then $x_1(b1, 0) = a$ for $b_1 > 0$:
  - If $x_1(b_1, 0) > a$, then buyer with $v_1 = b_1 > 0$ will bid 0^+ in case $v_1 = b_1$ and $v_2 = 0$. Its utility will improve from $v_1 - x_1(b_1, 0)$ to $v_1 - a$.
  - If $x_1(b_1, 0) < a$, then buyer with $v_1 = 0^+ > 0$ will bid b_1 in case $v_1 = b_1$ and $v_2 = 0$. Its utility will improve from $v_1 - a$ to $v_1 - x_1(b_1, 0)$.
- In general, $x_1(b_1, b_2) = x_1(b_2^+, b_2)$ for $b_1 > b_2$.
- Similarly, $x_2(b_1, b_2) = x_2(b_1, b_1^-)$ for $b_2 < b_1$.

Vickery's Impossibility:

- No protocol satisfies all four conditions of Ex-Post Efficiency, Budget balancedness, Truthfulness, Individual Rationality
- Proof: Let $b_1 >> b_2$, by bid independency,
  - $x_1(b_1, b_2) = x_1(b_2^+, b_2)$
  - $x_2(b_1, b_2) = x_2(b_1, b_1^-)$
- By Budget Balanced Condition,
  - $x_1(b_1, b_2) \geq x_2(b_1, b_2)$
  - $x_1(b_2^+, b_2) = x_1(b_1, b_2) \geq x_2(b_1, b_2) = x_2(b_1, b_1^-)$
  - implying $x_1(b_2^+, b_2) \geq x_2(b_1, b_1^-)$
    $\forall b_1, b_2 \in [\alpha_1, \beta_1] \cap [\alpha_2, \beta_2] : b_1 > b_2$
- $x_1(\alpha_1, \alpha_1^-) \geq x_2(\beta_2, \beta_2^-)$ which contradicts to IR $\alpha_1 \geq x_1(\alpha_1, \alpha_1^-)$ and $\beta_2^- \leq x(\beta_2, \beta_2^-)$, except where $\alpha_1 = \beta_2$ and the price is fixed a constant $\alpha_1 = \beta_2$.

Bayesian Double Auction (Myerson and Satterthwaite)

Input Model

- Participants: 1 buyer and 1 seller.
  - The value of the buyer is V_1.
  - The value of the seller is V_2.
- Common Knowledge: $\forall i = 1, 2$ V_i is a random variable with PDF $f_i(\cdot)$ continuous on $[\alpha_i, \beta_i]$. $f_i(\cdot) = F_i'(\cdot)$.
- Private Knowledge: Each knows its own value and the PDF of the other.
- Utility: Players are risk neutral and have additive separable utility for money and object.

Output Model

If the buyer bids b_1 and the seller bids b_2.

- Allocation: $p(b_1, b_2)$ for the probability the item is sold.
- Pricing: $x(b_1, b_2)$ for the expected payment from buyer to seller.
- Truthful: a truthful reporting mechanism is a BIC (Bayesian Incentive Compatible) if each maximizes its expected utility by reporting the truth.

Notations

- $\forall i = 1, 2,\ \bar{x}_i(v_i) = \int_{\alpha_i}^{\beta_i} x(v_i, t_{-i}) f_{-i}(t_{-i})\, dt_{-i}$
- $\forall i = 1, 2,\ \bar{p}_i(v_i) = \int_{\alpha_i}^{\beta_i} p(v_i, t_{-i}) f_{-i}(t_{-i})\, dt_{-i}$
- $U_1(v_1) = v_1 \bar{p}_1(v_1) - \bar{x}_1(v_1)$
- $U_2(v_2) = \bar{x}_2(v_2) - v_2 \bar{p}_2(v_2)$

Bayesian Incentive Compatibility (BIC)

- For the buyer, $\forall b_1 \in [\alpha_1, \beta_1]$: $U_1(v_1) \geq v_1 \bar{p}_1(b_1) - \bar{x}_1(b_1)$.
- For the seller, $\forall b_2 \in [\alpha_2, \beta_2]$: $U_2(v_2) \geq \bar{x}_2(b_2) - v_2 \bar{p}_2(b_2)$,

Definition of Individual Rationality (IR)

- $U_1(v_1) \geq 0$.
- $U_2(v_2) \geq 0$,

Objectives and Results

Define Ex Post Efficiency(EPE)

A mechanism (p, x) is ex post efficient iff

$$\begin{aligned} p(v_1, v_2) &= 1 \quad if \;\; v_1 > v_2 \\ &= 0 \quad otherwise \end{aligned} \tag{1}$$

Define Social Welfare

$$\begin{aligned}
&\int_{\alpha_1}^{\beta_1} U_1(v_1) f_1(v_1)\, dv_1 + \int_{\alpha_2}^{\beta_2} U_2(v_2) f_2(v_2)\, dv_2 \\
= \; &\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} (v_1 - v_2) p(v_1, v_2) f_1(v_1) f_2(v_2)\, dv_1\, dv_2 \qquad (2)
\end{aligned}$$

Define Market Maker's Revenue

$$U_0 = \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} [x_1(v_1, v_2) - x_2(v_1, v_2)] f_1(v_1) f_2(v_2)\, dv_1\, dv_2 \qquad (3)$$

Impossibility of Ex Post Efficiency

No mechanism exists in double auction which is simultaneously ex post efficient, Bayesian incentive compatible, budget balanced and individual rational.

Optimal Social Welfare

1. There is a truthful mechanism which maximizes the optimal social welfare of a double auction under Bayesian setting.
2. Note: there are exchange of money, with the requirement of IR and BIC but no EPE.

Optimal Market Maker's Revenue

1. There is a truthful mechanism which maximizes the market maker's revenue.
2. Note: Ex post efficiency is sacrisfied.

Characterizaction of BIC Condition

BIC and IR Characterization of Allocation

Let $p(\cdot,\cdot) : [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \rightarrow [0, 1]$. Then there exists a function $x(\cdot,\cdot)$ such that (p, x) is IR and BIC iff

$$\begin{aligned} & \bar{p}_1(\cdot) \ is \ non - decreasing \\ & \bar{p}_2(\cdot) \ is \ non - increasing \\ 0 \leq \quad & \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \{[v_1 - \tfrac{1-F_1(v_1)}{f_1(v_1)}] - [v_2 + \tfrac{F_2(v_2)}{f_2(v_2)}]\} \\ \times \quad & p(v_1, v_2) f_1(v_1) f_2(v_2)\, dv_1\, dv_2 \qquad (4) \end{aligned}$$

Base Case Utility

Base Case Utility

For any BIC mechanism,

$$U_1(\alpha_1) + U_2(\beta_2) = \min_{v_1} U_1(v_1) + \min_{v_2} U_2(v_2) =$$

$$= \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \{[v_1 - \frac{1 - F_1(v_1)}{f_1(v_1)}] - [v_2 + \frac{F_2(v_2)}{f_2(v_2)}]\}$$

$$\times p(v_1, v_2) f_1(v_1) f_2(v_2)\, dv_1\, dv_2$$

Total Utility under Ex Post Efficiency

Total Utility under Ex Post Efficiency

A mechanism (p, x) is ex post efficient iff $\alpha_1 < \beta_2$ and $\alpha_2 < \beta_1$. We derive

$$\begin{aligned} & U(\beta_2) + U(\alpha_1) \\ = & -\int_{\alpha_1}^{\beta_2} (1 - F_1(t)) F_2(t)\, dt \end{aligned} \tag{5}$$

Properties

Represent Base Case Utility

Utility Based on Accumulated Probability of Allocation

$$
\begin{aligned}
U_1(v_1) &= U_1(\alpha_1) + \int_{\alpha_1}^{v_1} \bar{p}_1(t_1)\, dt_1 \\
U_2(v_2) &= U_2(\beta_2) + \int_{v_2}^{\beta_2} \bar{p}_2(t_2)\, dt_2
\end{aligned}
\tag{6}
$$

1. Proof: Truth Telling vs Lying
   - Truth v_1: $U_1(v_1) = v_1 * \bar{p}_1(v_1) - \bar{x}(v_1) \geq v_1 * \bar{p}_1(v_1') - \bar{x}(v_1')$
   - Truth v_1': $U_1(v_1') = v_1' * \bar{p}_1(v_1') - \bar{x}(v_1') \geq v_1' * \bar{p}_1(v_1) - \bar{x}(v_1)$
   - $(v_1 - v_1')\bar{p}_1(v_1) \geq U_1(v_1) - U_1(v_1') \geq (v_1 - v_1')\bar{p}_1(v_1')$
   - $(v_1 - v_1') * [\bar{p}_1(v_1) - \bar{p}_1(v_1')] \geq 0$, $\bar{p}_1(\cdot)$ are non-decreasing.
   - $\frac{\partial U_1(v_1)}{\partial v} = \bar{p}_1(v_1)$ and $\frac{\partial U_2(v_2)}{\partial v} = -\bar{p}_2(v_2)$
2. End of Proof: Integration starting at the base case (α_1, β_2).

Expected Total Utility

By Definition of utility functions of buyer and seller,

$$\begin{aligned} & \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}(v_1 - v_2)p(v_1, v_2)f_1(v_1)f_2(v_2)\,dv_1\,dv_2 \\ = \quad & \int_{\alpha_1}^{\beta_1} U_1(v_1)f_1(v_1)\,dv_1 + \int_{\alpha_2}^{\beta_2} U_2(v_2)f_2(v_2)\,dv_2 \end{aligned} \tag{7}$$

Total Utility = Base Case + Allocation Probability

By separation of utility function (6),

$$\begin{aligned} & \int_{\alpha_1}^{\beta_1} U_1(v_1) f_1(v_1)\, dv_1 + \int_{\alpha_2}^{\beta_2} U_2(v_2) f_2(v_2)\, dv_2 \\ = \quad & U_1(\alpha_1) + \int_{\alpha_1}^{\beta_1} \int_{\alpha_1}^{v_1} \bar{p}_1(t_1)\, dt_1 f_1(v_1)\, dv_1 \\ + \quad & U_2(\beta_2) + \int_{\alpha_2}^{\beta_2} \int_{v_2}^{\beta_2} \bar{p}_2(t_2)\, dt_2 f_2(v_2)\, dv_2 \end{aligned} \tag{8}$$

Initial Utility

Exchange order of integrations,

$$\begin{aligned}
& U_1(\alpha_1) + \int_{\alpha_1}^{\beta_1} \int_{\alpha_1}^{v_1} \bar{p}_1(t_1)\, dt_1\, f_1(v_1)\, dv_1 \\
+ \quad & U_2(\beta_2) + \int_{\alpha_2}^{\beta_2} \int_{v_2}^{\beta_2} \bar{p}_2(t_2)\, dt_2\, f_2(v_2)\, dv_2 \\
= \quad & U_1(\alpha_1) + U_2(\beta_2) \\
+ \quad & \int_{\alpha_1}^{\beta_1} [1 - F_1(t_1)]\bar{p}_1(t_1)\, dt_1 + \int_{\alpha_2}^{\beta_2} F_2(t_2)\bar{p}_2(t_2)\, dt_2
\end{aligned} \tag{9}$$

Initial Utility

Expand the integrations,

$$\int_{\alpha_1}^{\beta_1}[1 - F_1(t_1)]\bar{p}_1(t_1)\,dt_1 + \int_{\alpha_2}^{\beta_2} F_2(t_2)\bar{p}_2(t_2)\,dt_2$$
$$= \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\{[1 - F_1(v_1)]f_2(t_2) + F_2(v_2)f_1(t_1)\}p(t_1, t_2)\,dt_1\,dt_2 \quad (10)$$

Initial Utility

In summary,

$$
\begin{aligned}
& U_1(\alpha_1) + U_2(\beta_2) \\
+ \quad & \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \{[1 - F_1(v_1)] f_2(v_2) + F_2(v_2) f_1(v_1)\} p(v_1, v_2)\, dv_1\, dv_2 \\
= \quad & \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} (v_1 - v_2) p(v_1, v_2) f_1(v_1) f_2(v_2)\, dv_1\, dv_2
\end{aligned}
$$

End of Proof on Initial Utility Conditions

Expand the integrations,

$$U_1(\alpha_1) + U_2(\beta_2)$$
$$= \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \{[v_1 - \frac{1-F_1(v_1)}{f_1(v_1)}] - [v_2 + \frac{F_2(v_2)}{f_2(v_2)}]\} p(v_1, v_2) f_1(v_1) f_2(v_2) dv_1 dv_2 \tag{11}$$

BIC Mechanism Characterization

BIC and IR Characterization

Let $p(\cdot,\cdot) : [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \to [0,1]$. Then there exists a function $x(\cdot,\cdot)$ such that (p, x) is IR and BIC iff

$$\bar{p}_1(\cdot) \ \textit{is non} - \textit{decreasing}$$

$$\bar{p}_2(\cdot) \ \textit{is non} - \textit{increasing}$$

$$\begin{aligned} 0 \leq \quad & \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \{[v_1 - \frac{1-F_1(v_1)}{f_1(v_1)}] - [v_2 + \frac{F_2(v_2)}{f_2(v_2)}]\} \\ \times \quad & p(v_1, v_2) f_1(v_1) f_2(v_2)\, dv_1\, dv_2 \end{aligned} \tag{12}$$

Necessity of Characterization

- $\bar{p}_i(\cdot)$ is monotone by BIC.
- Formula (12) holds as the sum of the initial utilities is non-zero.

Sufficiency of Characterization

- Define x:

$$x(v_1, v_2) = \int_{\alpha_1}^{v_1} t_1 \, d[\bar{p}_1(t_1)] - \int_{\alpha_2}^{v_2} t_2 \, d[-\bar{p}_2(t_2)]$$
$$+\alpha_1 \bar{p}_1(\alpha_1) + \int_{\alpha_2}^{\beta_2} t_2[1 - F_2(t_2)] d[-\bar{p}_2(t_2)] \qquad (13)$$

.

- x is similar to that of optimal auction design, with the last two terms determined by $U_1(\alpha_1) = 0$.
- It is a matter of checking to see (p, x) is IR and BIC, as follows.

Compute $U_1(\alpha_1)$.

$$U_1(\alpha_1) = \alpha_1 \bar{p}_1(\alpha_1) - \int_{\alpha_2}^{\beta_2} x(\alpha_1, v_2) f_2(v_2)\, dv_2$$

$$\begin{aligned} x(\alpha_1, v_2) = & \quad \textstyle\int_{\alpha_1}^{\alpha_1} t_1\, d[\bar{p}_1(t_1)] - \int_{\alpha_2}^{v_2} t_2\, d[-\bar{p}_2(t_2)] \\ & +\alpha_1 \bar{p}_1(\alpha_1) + \textstyle\int_{\alpha_2}^{\beta_2} t_2[1 - F_2(t_2)] d[-\bar{p}_2(t_2)] \\ = & \quad -\textstyle\int_{\alpha_2}^{v_2} t_2\, d[-\bar{p}_2(t_2)] \\ & +\alpha_1 \bar{p}_1(\alpha_1) + \textstyle\int_{\alpha_2}^{\beta_2} t_2[1 - F_2(t_2)] d[-\bar{p}_2(t_2)] \qquad (14) \end{aligned}$$

.

$U_1(\alpha_1) = 0$.

$$U_1(\alpha_1) = \int_{\alpha_2}^{\beta_2}\int_{\alpha_2}^{v_2} t_2\, d[-\bar{p}_2(t_2)] f_2(v_2)\, dv_2 - \int_{\alpha_2}^{\beta_2} t_2[1 - F_2(t_2)] d[-\bar{p}_2(t_2)]$$

which is zero since
$\int_{\alpha_2}^{\beta_2}\int_{\alpha_2}^{v_2} t_2\, d[-\bar{p}_2(t_2)] f_2(v_2)\, dv_2 = \int_{\alpha_2}^{\beta_2}[\int_{t_2}^{\beta_2} f_2(v_2)\, dv_2] t_2\, d[-\bar{p}_2(t_2)]$,
which is the same as the 2nd term in the representation of $U_1(\alpha_1)$.

Buyer is BIC.

Consider the difference in utility of the buyer in reporting truth v_1 or lying to report b_1:

$$\begin{aligned} & (v_1\bar{p}_1(v_1) - v_1\bar{p}_1(b_1) - (\bar{x}_1(v_1) - \bar{x}_1(b_1)) \\ = & \quad v_1 \int_{b_1}^{v_1} d[\bar{p}_1(t_1)] - \int_{b_1}^{v_1} t_1 \, d[\bar{p}_1(t_1)] \\ = & \quad \int_{b_1}^{v_1} (v_1 - t_1) \, d[\bar{p}_1(t_1)] \\ \geq & \quad 0, \end{aligned} \tag{15}$$

by monotonicity property of $\bar{p}_1(t_1)$

Conclusion of BIC and IR

Seller's utility is also optimized by reporting truth. IR of the buyer follows by $U_1(\alpha_1) = 0$ and monotonicity of $\bar{p}_1(t_1)$. IR of the seller is by the assumed property [4].

Ex-Post Efficiency

Key Observation

A mechanism (p, x) is ex post efficient iff $\alpha_1 < \beta_2$ and $\alpha_2 < \beta_1$. We derive

$$\begin{aligned} & U(\beta_2) + U(\alpha_1) \\ = \quad & -\int_{\alpha_1}^{\beta_2} (1 - F_1(t)) F_2(t)\, dt \end{aligned} \tag{16}$$

which violates IR. Therefore, it requires a subsidy be provided from an outside party to make it to exist.

Maximization of Social Welfare

Total Gains

$$
\begin{aligned}
& \int_{\alpha_1}^{\beta_1} U_1(v_1) f_1(v_1)\, dv_1 + \int_{\alpha_2}^{\beta_2} U_2(v_2) f_2(v_2)\, dv_2 \\
= \quad & \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} (v_1 - v_2) p(v_1, v_2) f_1(v_1) f_2(v_2)\, dv_1\, dv_2
\end{aligned} \tag{17}
$$

Note: The market objective is to optimize the total social welfare but the buyer and the seller want to maximise their own utility. The question is whether the market can coordinate the two goals.

Virtual Values

$$\begin{aligned} c_1(v_1,\alpha) &= v_1 - \alpha\frac{1-F_1(v_1)}{f_1(v_1)} \\ c_2(v_2,\alpha) &= v_2 + \alpha\frac{F_2(v_2)}{f_2(v_2)} \end{aligned} \tag{18}$$

$$\begin{aligned} p^{\alpha}(v_1,v_2) &= 1 \quad if \quad c_1(v_1,\alpha) \geq c_2(v_2,\alpha) \\ &= 0 \quad if \quad c_1(v_1,\alpha) < c_2(v_2,\alpha) \end{aligned}$$

A Truthful Mechanism Maximizes Total Gains

1. If an BIC mechanism (p, x) satisfies $U_1(\alpha_1) = U_2(\beta_2) = 0$ and $p = p^\alpha$ for some $\alpha \in [0, 1]$, then it maximizes the total gains among all BIC and IR mechanisms.
2. If $c_i(\cdot, 1)$ is increasing in $[\alpha_i, \beta_i]$ $(\forall i = 1, 2)$ and if the intersection of the interiors of the two intervals is non-empty, then such a mechanism exists.

The Optimization Model

The goal here is to choose $p(b_1, b_2)$ to maximize the social welfare, subject to the IR and BIC condition.

$$\begin{aligned} \max \quad & \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} (v_1 - v_2) p(v_1, v_2) f_2(v_2) f_1(v_1)\, dv_1\, dv_2 \\ s.t. \quad & \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} [c_1(v_1, 1) - c_2(v_2, 1)] p(v_1, v_2) f_1(v_1) f_2(v_2)\, dv_1\, dv_2 \geq 0 \end{aligned} \tag{19}$$

The truthful condition (4) implies that $p_1(v_1)$ and $p_2(v_2)$ be increasing and decreasing, respectively. The condition $U_1(\alpha_1) = U_2(\beta_2) = 0$ implies Condition (19), and hence IR holds.

Lagrangian of The Optimal Gain

$$
\begin{aligned}
& \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}[v_1 + \lambda c_1(v_1, 1) - v_2 - \lambda c_2(v_2, 1)] \\
& \quad p(v_1, v_2) f_2(v_2) f_1(v_1)\, dv_1\, dv_2 \\
= & \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}[v_1 + \lambda(v_1 - \tfrac{1-F_1(v_1)}{f_1(v_2)}) - v_2 - \lambda(v_2 + \tfrac{F_2(v_2)}{f_2(v_2)})] \\
& \quad p(v_1, v_2) f_2(v_2) f_1(v_1)\, dv_1\, dv_2 \\
= & (1+\lambda)\int\int[c_1(v_1, \tfrac{\lambda}{1+\lambda}) - c_2(v_2, \tfrac{\lambda}{1+\lambda})] \\
& \quad p(v_1, v_2) f_2(v_2) f_1(v_1)\, dv_1\, dv_2
\end{aligned}
$$

Optimality in the Theorem

1. $p^{\alpha}(v_1, v_2)$ clearly maximizes the Lagrangian at $\alpha = \frac{\lambda}{1+\lambda}$.

Existence Condition for truthful p and $U_1(\alpha_1) = U_2(\beta_2) = 0$

1. If $c_1(\cdot, 1)$ and $c_2(\cdot, 1)$ increase on $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$, respectively
2. so do $c_i(\cdot, \alpha)$ $\forall \alpha \in [0, 1]$.
3. $p^\alpha(v_1, v_2)$ increases in v_1 and decreases in v_2.
4. $\bar{p}_1^\alpha(\cdot)$ is increasing and $\bar{p}_2^\alpha$ decreasing.

Define $G(\cdot)$

$$G(\alpha) = \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} [c_1(v_1, 1) - c_2(v_2, 1)] p^{\alpha}(v_1, v_2) f_1(v_1) f_2(v_2)\, dv_1\, dv_2$$

- We want to prove $G(\alpha) = 0$ for some α to prove the truthful condition for some (p, x).
- Clearly, $G(1) \geq 0$ by definition: $p(v_1, v_2) = 1$ if and only if $c_1(v_1, 1) \geq c_2(v_2, 1)$.

$c_i(v_i, \alpha)$ decreasing in α

$$c_1(v_1, \alpha) - c_2(v_2, \alpha) = \quad (v_1 - v_2) - \alpha\left(\frac{1-F_1(v_1)}{f_1(v_1)} + \frac{F_2(v_2)}{f_2(v_2)}\right)$$

which is decreasing in α.

So is $p^\alpha(v_1, v_2)$.

Change in $G(\alpha)$

$$G(\alpha) = \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} [c_1(v_1,1) - c_2(v_2,1)]p^{\alpha}(v_1,v_2)f_1(v_1)f_2(v_2)\,dv_1\,dv_2$$

$G(\beta)$ changes from $G(\alpha)$, where $\beta > \alpha$, only where $0 = p^{\beta}(v_1,v_2) < p^{\alpha}(v_1,v_2) = 1$, in which case, we also have $c_1(v_1,\beta) < c_2(v_2,\beta)$ and so $c_1(v_1,1) < c_2(v_2,1)$. Therefore, $[c_1(v_1,1) - c_2(v_2,1)]p^{\beta}(v_1,v_2)f_1(v_1)f_2(v_2) \geq [c_1(v_1,1) - c_2(v_2,1)]p^{\alpha}(v_1,v_2)f_1(v_1)f_2(v_2)$ for all $(v_1, v_2, \alpha < \beta)$

$G(\alpha)$ increases in α.

$G(\cdot)$ is continuous

$$G(\alpha) = \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} [c_1(v_1, 1) - c_2(v_2, 1)] p^\alpha(v_1, v_2) f_1(v_1) f_2(v_2)\, dv_1\, dv_2$$

Fixing α and v_1, $c_1(v_1, \alpha) = c_2(v_2, \alpha)$ at at most one value v_2, denoted by $g(v_1, \alpha)$, which is continuous in v_1 and α.

$$G(\alpha) = \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{g(v_1,\alpha)} [c_1(v_1, 1) - c_2(v_2, 1)] p^\alpha(v_1, v_2) f_1(v_1) f_2$$

which implies $G(\alpha)$ continuous in α.

$G(\alpha) = 0$ has a solution

$$G(0) = \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} [c_1(v_1, 1) - c_2(v_2, 1)] p^0(v_1, v_2) f_1(v_1) f_2(v_2) \, dv_1 \, dv_2$$

$p^0(\cdot)$ is an Ex Post efficient, IR, BIC,

$G(0)$ can only be < 0 or a contradiction to Myerson-Satterthwaite impossibility theorem.

$G(\alpha)$

$$G(0) = \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} [c_1(v_1, 1) - c_2(v_2, 1)] p^0(v_1, v_2) f_1(v_1) f_2(v_2) \, dv_1 \, dv_2$$

$p^0(\cdot)$ is an Ex Post efficient, IR, BIC,

which can only be < 0 or a contradiction to Myerson-Satterthwaite impossibility theorem.

$p^\alpha(\cdot)$ is the required solution of the theorem.

Trade Through a Broker

Input Model

- Participants: 1 buyer, 1 seller, and a broker.
    - The value of the buyer is V_1.
    - The value of the seller is V_2.
- Common Knowledge: $\forall i = 1, 2$ V_i is a random variable with PDF $f_i(\cdot)$ continuous on $[\alpha_i, \beta_i]$. $f_i(\cdot) = F_i'(\cdot)$.
- Private Knowledge: Each knows its own value and the PDF of the other. The broker knows only the PDF of the buyer and the seller.
- Utility: Players are risk neutral and have additive separable utility for money and object. The broker is interested in money only.

Output Model

If the buyer bids b_1 and the seller bids b_2.

- Allocation: $p(b_1, b_2)$ for the probability the item is sold.
- Pricing: $x1(b_1, b_2)$ for the expected payment from buyer to broker, $x_2(b_1, b_2)$ the received payment from the broker to the seller.
- Truthful: a truthful reporting mechanism is a BIC (Bayesian Incentive Compatible) if each maximizes its expected utility by reporting the truth.

Notations

- $\forall i = 1, 2,\ \bar{x}_i(v_i) = \int_{\alpha_i}^{\beta_i} x_i(v_i, t_{-i}) f_{-i}(t_{-i})\, dt_{-i}$
- $\forall i = 1, 2,\ \bar{p}_i(v_i) = \int_{\alpha_i}^{\beta_i} p(v_i, t_{-i}) f_{-i}(t_{-i})\, dt_{-i}$
- $U_1(v_1) = v_1 \bar{p}_1(v_1) - \bar{x}_1(v_1)$
- $U_2(v_2) = \bar{x}_2(v_2) - v_2 \bar{p}_2(v_2)$
- $U_0 = \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} [x_1(t_1, t_2) - x_2(t_1, t_2)] f_1(t_1) f_2(t_2)\, dt_1\, dt_2.$

Bayesian Incentive Compatibility (BIC)

- For the buyer, $\forall b_1 \in [\alpha_1, \beta_1]$: $U_1(v_1) \geq v_1 \bar{p}_1(b_1) - \bar{x}_1(b_1)$.
- For the seller, $\forall b_2 \in [\alpha_2, \beta_2]$: $U_2(v_2) \geq \bar{x}_2(b_2) - v_2 \bar{p}_2(b_2)$,

Definition of Individual Rationality (IR)

- $U_1(v_1) \geq 0.$
- $U_2(v_2) \geq 0,$

Virtual Values

$$
\begin{aligned}
C_1(v_1, \alpha) &= v_1 - \alpha \frac{1 - F_1(v_1)}{f_1(v_1)} \\
C_2(v_2, \alpha) &= v_2 + \alpha \frac{F_2(v_2)}{f_2(v_2)}
\end{aligned} \tag{20}
$$

Truthful Double Auction with an Agent

For any BIC with a broker, $\bar{p}_1(\cdot)$ is weakly increasing and $\bar{p}_2(\cdot)$ is weakly decreasing, and

$$\begin{aligned} & U_0 + U_1(\alpha_1) + U_2(\beta_2) \\ = \quad & U_0 + min_{v_1 \in [\alpha_1, \beta_1]} U_1(v_1) + \min_{v_2 \in [\alpha_2, \beta_2]} U_2(v_2) \\ = \quad & \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} [C_1(v_1) - C_2(v_2)] p(v_1, v_2) f_1(v_1) f_2(v_2) \, dv_1 \, dv_2. \quad (21) \end{aligned}$$

Sketch of Derivation

Similar to (7), we have

$$\begin{aligned} &\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}(v_1 - v_2)p(v_1, v_2)f_1(v_1)f_2(v_2)\,dv_1\,dv_2 \\ = &\int_{\alpha_1}^{\beta_1} U_1(v_1)f_1(v_1)\,dv_1 + \int_{\alpha_2}^{\beta_2} U_2(v_2)f_2(v_2)\,dv_2 + U_0, \end{aligned}$$

which derives

$$\begin{aligned} &U_0 + U_1(\alpha_1) + U_2(\beta_2) \\ = &\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\{[v_1 - \tfrac{1-F_1(v_1)}{f_1(v_1)}] - [v_2 + \tfrac{F_2(v_2)}{f_2(v_2)}]\}p(v_1, v_2)f_1(v_1)f_2(v_2)dv_1dv_2 \end{aligned} \tag{22}$$

Ex Post Efficiency with A Broker

Under Ex Post Efficiency, the allocation is fixed

$$\begin{aligned} p(v_1, v_2) &= 1 \quad if \;\; v_1 > v_2 \\ &= 0 \quad if \;\; v_1 < v_2 \end{aligned} \tag{23}$$

We then place the EPE allocation into the following integration to verify if it is non-negative.

$$\begin{aligned} & U_0 + U_1(\alpha_1) + U_2(\beta_2) \\ = \quad & \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \{[v_1 - \tfrac{1-F_1(v_1)}{f_1(v_1)}] - [v_2 + \tfrac{F_2(v_2)}{f_2(v_2)}]\} p(v_1, v_2) f_1(v_1) f_2(v_2) dv_1 dv_2 \end{aligned} \tag{24}$$

Case 1: $\alpha_1 < \alpha_2 < \beta_1 < \beta_2$

$$\begin{aligned}
& U_0 + U_1(\alpha_1) + U_2(\beta_2) \\
= & \int_{\alpha_2}^{\beta_1} [\int_{\alpha_2}^{v_1} \{v_1 f_1(v_1) + F_1(v_1) - 1\} f_2(v_2)\, dv_2]\, dv_1 \\
& - \int_{\alpha_2}^{\beta_1} [\int_{\alpha_2}^{v_1} \{v_2 f_2(v_2) + F_2(v_2)\}\, dv_2] f_1(v_1)\, dv_1 \\
= & \int_{\alpha_2}^{\beta_1} \{v_1 f_1(v_1) + F_1(v_1) - 1\} F_2(v_1)\, dv_1 \\
& - \int_{\alpha_2}^{\beta_1} v_1 F_2(v_1) f_1(v_1)\, dv_1 \\
= & - \int_{\alpha_2}^{\beta_1} \{1 - F_1(v_1)\} F_2(v_1)\, dv_1
\end{aligned} \tag{25}$$

Impossibility Result with A Broker: IR, BIC, Budget Balancedness, EPE

Under Ex Post Efficiency, we have

$$\begin{aligned} & U_0 + U_1(\alpha_1) + U_2(\beta_2) \\ = & -\int_{\alpha_2}^{\beta_1} \{1 - F_1(v_1)\} F_2(v_1)\, dv_1 \end{aligned} \tag{26}$$

which violates IR.

Maximizing Revenue of the Broker

- If $C_i(\cdot)$, $i = 1, 2$, are monotone increasing on $[\alpha_i, \beta_i]$, among all BIC and IR mechanisms, the broker's revenue is maximized by a mechanism that transfers the item if and only if $C_1(\tilde{V}_1) \geq C_2(\tilde{V}_2)$.
- where $U_0 = \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} [C_1(t_1) - C_2(t_2)] f_1(t_1) f_2(t_2) \, dt_1 \, dt_2 - U_1(\alpha_1) - U_2(\beta_2)$

Allocation

$$\begin{aligned} p(v_1, v_2) &= 1 \quad \textit{if} \;\; C_1(v_1) \geq C_2(v_2) \\ &= 0 \quad \textit{if} \;\; C_1(v_1) < C_2(v_2) \end{aligned} \tag{27}$$

Pricing

$$\begin{aligned} x_1(v_1, v_2) &= p(v_1, v_2) \cdot \min\{t_1 : t_1 \geq \alpha_1 \& C_1(t_1) \geq C_2(v_2)\} \\ x_2(v_1, v_2) &= p(v_1, v_2) \cdot \max\{t_2 : t_2 \leq \beta_2 \& C_2(t_1) \leq C_1(v_1)\} \end{aligned} \tag{28}$$

Proof

- Miss report will not change one's payment but reduce its winning probability
- $U_1(\alpha_1) = U_2(\beta_2) = 0$ since
  1. $x_1(v_2, \alpha_1) = p(v_2, \alpha_1)\alpha_1$
  2. $x_2(\beta_2, v_1) = p(\beta_2, v_1)\beta_2$

Further readings

- R.B. Myerson, M.A. Satterthwaite, Efficient mechanisms for bilateral trading, J. Econom. Theory 29 (2) (1983) 265–281.
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Revenue maximization in a Bayesian double auction market