Digital Goods Pricing

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1. Digital Goods
2. Competitive Auction

Properties to Establish

- There are n Buyers $V = \{v_1 \geq v_2 \cdots \geq v_n\}$ and an infinite number of copies of the same goods from a single seller.
- Consider three different types of pricing models
  - Fixed Price: a price is announced ahead of time, everyone with utility ≥ 0 pays the fixed price.
  - Fixed Revenue: The revenue R is given as an input. The price is set to the minimum p such that $p|\{b_i \geq p : i = 1, 2, \cdots, n\}| \geq R$.
  - Fixed number of winners: price p sets to be the $(k+1)$-st highest bid and make anyone bidding higher than p a winner.
- Prove they are truthful or else give a counter example.

Digital Goods

Modeling Digital Goods

- There are n Buyers
  $V = \{v_1 = 11, v_2 = 7, v_3 = 5, v_4 = 3, v_5 = 2, v_6 = 1\}$ and an infinite number of goods of the same goods from a single seller.
- Pricing Models
  - Fixed Price: a price is announced ahead of time, everyone with utility ≥ 0 pays the fixed price.
  - Fixed Revenue
  - Fixed number of winners

Fixed Revenue

- A target revenue R is decided first.
- Find how many buyers to sell the item to.
  - $\nu = \max\arg\{i : i * b_i \geq R, i \in V\}$
  - Sell the item to the top ν bidders and charge them $\frac{R}{\nu}$ each for a total of R.
- Truthful proof: If a winner remains a winner, the results does not change. It would not want to be a loser. A loser can only bid higher to win, in which case it would have to pay more than its value.
- It is possible that no one wins and the seller sells nothing.

Fixed Number of Winners

- Vickrey Auction: sell one item.
- k-Vickrey: Sell k items to k highest bidders and charge them $(k+1)$-st bid.
- Truthful proof: trivial.

Competitive Auction

Seller's Optimum

- The maximum possible single price revenue: $\mathcal{F} = \max\{iv_i : v_i \in V\}$
- It is impossible to have a protocol that always gets $\mathcal{F}$ or approximate it. Proven next ppt. Main ideas:
  - A truthful protocol is bid-independent.
  - Consider two buyers with v_1, v_2. There are four functions $f_1(\cdot), f_2(\cdot), g1(\cdot), g2(\cdot)$ such that v_1 bids $b_1 = g_1(v_1)$ and pays $f_1(b_2)$ and v_2 bid $b_2 = g_2(v_2)$ and pays $f_2(b_1)$ if it wins.
- We should approximate $\mathcal{F}^{(2)} = \max\{iv_i : v_i \in V, i \geq 2\}$

Winner Determination Functions

- Consider two buyers with $v_1 > v_2$.
- Let $b_i = g_i(v_i)$ be the bidding function of buyer $i = 1, 2$.
- Let $w(b_1, b_2) \subseteq \{1, 2\}$ be the function selecting a winner.
- Let $f_i(b_1, b_2)$ be the price functions that the winner i, $i = 1, 2$, pays.

Bid-independence and threshold price function

1. If $1 \in w(b_1, b_2)$ then $b_1 \geq f_1(b_1, b_2)$. (individual rationality)
2. If $1 \in w(b_1, b_2)$ then $\forall b_1' > b_1 : 1 \in w(b_1', b_2)$. (truthfulness for $v_1 = b_1'$)
3. If $1 \in w(b_1, b_2)$ then $\forall b_1' > b_1 : f_1(b_1', b_2) = f_1(b_1, b_2)$. (truthfulness)
4. There is a minimum b_1^* such that $\forall b_1 > b_1^* : 1 \in w(b_1, b_2)$ and $\forall b_1 < b_1^* : 1 \notin w(b_1, b_2)$
   - Define $b_1^* = \inf\{b_1 : 1 \in w(b_1, b_2)\}$. It is well defined by the condition $b_1 \geq 0$.
   - $\forall b_1 > b_1^*, f_1(b_1, b_2) = b_1^*$ by item 2 and 3 above.
   - Similarly $\forall b_1 < b_1^* : 1 \notin w(b_1, b_2)$ and $f_1(b_1, b_2) < b_1$. It is equivalent to set $f_1(b_1, b_2) = b_1^*$.
5. It generalises: consider the case b_2 being a vector.

Impossibility to get Seller's Optimum

- We consider the case $v_1 = v >> v_2 = 1$.
- Since v_1 pays $f_1(v_2) = f_1(1)$, the revenue from v_1 is $f_1(1)$ when v_1 is sufficiently large.
- v_2 will not pay more than 1 since $v_2 = 1$.
- Therefore, $R =\leq 1 + f_1(1)$.
- $\mathcal{F} \geq v_1 >> R$.
- Further, it is impossible to obtain a constant fraction of $\mathcal{F}$.

Competitive Ratio

It involves in two concepts:

- The offline optimum: Knowing all the information of buyers, what is the optimal revenue the seller can collect.
    1. distinguishing prices where each buyer pays its own value.
    2. m winner single price optimum $\mathcal{F}^{(m)}(V) = \max\{iv_i : v_i \in V, i \geq m\}$.
- The revenue obtained by the pricing protocol A $R(A, V)$ on the set V.
- Competitive Ratio: $r^{(m)} = \sup_A \inf_V \frac{R(A,V)}{\mathcal{F}^{(m)}(V)}$.
- The previous result shows that $r^{(1)}$ is unbounded. We are interested in $r^{(2)}$.

Random Sampling

1. Estimate the prices:
   - split the buyers' bids into two groups G_1 and G_2, by randomly ((1/2,1/2) probability) placing a buyer.
   - Find the optimum price p_1 and p_2 for the social optima, r_1 and r_2 of groups G_1 and G_2.
2. Set the prices:
   1. Fixed Price: Set the price p_2 for G_1 and p_1 for G_2.
   2. Fixed Revenue: Set the target revenue r_2 for G_1, and r_1 for G_2.

Truthfulness of the random sampling algorithm

Everyone's paying price is his own bid independent.

- The price for buyers $G1$ is determined by bids of $G2$.
- The price for buyers $G2$ is determined by bids of $G1$.
- Therefore, no one can pay less by changing its own bid.

Closeness to $\mathcal{F}^{(2)}$ of the random sampling algorithm

Main ideas

- Let p be the winning price of $\mathcal{F}^{(2)}$. W the corresponding winners. Let $v_j = p$.
- Roughly half the winners of $\mathcal{F}^{(2)}$ are in the same group G_1 or G_2 as v_j does, half of them are in the other group.
- $p_2 * |W2| \geq \frac{1}{2} p|W|$.
- For each price p' roughly half of those larger than it is in $G1$: $W2' = \{t : b_t > p_2, t \in G1\}$ in size is roughly the same as $W2$.
- Therefore $p_2 * |W2'|$ is roughly $\frac{1}{2} p|W|$.

Outline of Formal Proof

Consider Fixed Revenue

- Consider the (unknown) optimal price p^* for $\mathcal{F}^{(2)}$ and the number k of the winners W.
- Let $W_i = W \cap G_i$ and $k_i = |W_i|$, $i = 1, 2$.
- Then $k_1 + k_2 = k$ and $r^{(i)} \geq p^* k_i$, $i = 1, 2$.
- Given k_1, k_2, the revenue is therefore at least as much as $\min\{k_1, k_2\} p^*$
- The expected revenue is, hence, at least

$$p^* \sum_{k_1=1}^{k-1} \min\{k_1, k - k_1\} \binom{k}{k_1} 2^{-k}$$

Competitive Ratio of Four

Use the Fixed Revenue protocol. Competitive ratio is

$$\frac{1}{k}\sum_{k_1=1}^{k-1}\min\{k_1, k-k_1\}\binom{k}{k_1}2^{-k},$$

which reduces to $\frac{1}{2}-2^{-k}\binom{k-1}{\kappa}$ (proven next ppt), for $k=2\kappa$, or $k=2\kappa+1$. It achieves the minimum when $k=2$ or $k=3$.

Computing $\sum_{k_1=1}^{k-1} \min\{k_1, k-k_1\}\binom{k}{k_1}$

When $k = 2\kappa$, or $k = 2\kappa + 1$,

$$\sum_{k_1=1}^{k-1} \min\{k_1, k-k_1\}\binom{k}{k_1} = \sum_{k_1=1}^{\kappa} k_1\binom{k}{k_1} + \sum_{k_1=1+\kappa}^{k-1} (k-k_1)\binom{k}{k_1}$$

$$RHS = k * \sum_{k_1-1=0}^{\kappa-1} \binom{k-1}{k_1-1} + k * \sum_{k_1=1+\kappa}^{k-1} \binom{k-1}{t}$$

$$RHS = k*\sum_{t=0}^{\kappa-1} \binom{k-1}{t} + k*\sum_{t=1+\kappa}^{k-1} \binom{k-1}{t} = k*[2^{k-1} - \binom{k-1}{\kappa}]$$

Exercise

- What is the best approximation ratio for digital goods auction ?
- What is the result if you know the distribution of private values follows power law?
  - The powerlaw distribution has a density function $f(x) = x^{-\alpha}$, $\alpha > 2$.
- Suppose what is the optimal strategy if you sell your digital goods in two days?