Sponsored Search Market

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Sponsored Search Market

The Practical Problem

1. Place Ads along with Search Results
   - Search is a strong signal of interests from the SE user.
   - It offers an opportunity to place a suitable ad.
   - It improves efficiency from email ads.
2. How to allocation ads and how to price them?

Model of Input Data: $(M, q; N, v)$

1. There is a set M of m items/slots for sale
   - Ordered by quality

$$q_1 \geq q_2 \geq \cdots \geq q_m \geq 0$$

   - q_i measures how much attention the slot i receives from a search engine user , such as related to the click through rate (CTR), or to the impression.
2. There is a set N of n advertisers who buy the slots.
   - Order them by the value per quality

$$v_1 \geq v_2 \geq \cdots \geq v_n$$

   - Values are private information only known by the owners

Simplification

1. $m \leq n$, by introducing extra dummy buyers.
2. When there is no ambiguity,
   - Use v_i or $i \in N$ for buyers
   - Use q_j or $j \in M$ for items/slots
3. The value of the item j assigned to a buyer i will be $v_i * q_j$.

NOTE: The model makes optimal matching easy to obtain which is not always possible and is itself an important research problem. But here the assumption makes us focus on the economic issues involved.

Notation: price and allocation (p, x)

- Allocation: $y : M \to N$.
  - Each item is sold to one buyer.
    - $\forall j \in M$ let y_j be the buyer who gets item j.
    - $y_j \neq y_{j'}$ for $j \neq j'$.
  - Each buyer gets one item or none.
    - $x : N \to M$
    - $x_i \in M$ or x_i is unassigned ($x_i = \emptyset$).
    - $x_i \neq x_{i'}$ if $x_i \neq \emptyset$.
  - The inverse function: $x_{y_j} = j$ and $y_{x_i} = i$ if $x_i \neq \emptyset$.
- Pricing:
  - Price of item j: $p_j \geq 0, \forall j \in M$.

Utilities of Pariticipating Agents

- Buyers: Value minus cost
  - $\forall i \in N : u_i(v_i, q, x) = v_i * q_{x(i)} - p_{x(i)}$
- The Seller: Total income
  - Revenue: $R = \sum_{j \in M} p_j$.
- Social Welfare: Total value
  - $SW = \sum_{j \in M} v_j q_j$.

Views of Pariticipating Agents

- Buyers: Its own private value v_i
  - Its submitting bid b_i which does not have to be the same as v_i
  - It knows the protocol of the seller
  - Buyer choosees what to submit to maximize its own utility, according to Seller's protocol.
- The Seller: may not know values of buyers.
  - wants to maximise its total income
  - has to design a protocol to determine how to sell its slots.

Market Making

1. Decide on allocation and pricing (p, x) to maximize
   - Revenue
     - To create profit for the market maker
   - Social welfare?
     - To encourage participation
2. Information assymetry: Buyers' utilities may not be fully known to the market designer.
3. Three levels of knowledge:
   - Private information
   - Bayesian information: Probability distribution
   - Common Knowledge

VCG in Sponsored Search Market

Market Design Principles

- Individual Rationaliy: Each buyer maximizes its own utility
- Global Constraints:
  - Truthful.
    - It is a dominating strategy for a buyer to report its true value.
    - Example: The second price auction
  - Market clearance.
    - All goods are sold or else priced at zero.
    - Example: Market equilibrium
  - Covet-free
    - No buyer would prefer something else to its own allocation at current price.

VCG Auction Protocol(Vickrey'61, Clarke'71, Groves'73)

Allocation: Find an allocation (a matching) $MT(M, q; N, v)$ of size m with the maximum social welfare.

- Value of item j for Buyer i: $v_i * q_j$.
- Sort $v_1 \geq v_2 \geq \cdots \geq v_n$, $q_1 \geq q_2 \geq \cdots \geq q_m$
- The maximum weighted matching $MT = \{(v_j, q_j) : j \in M\}$ with total weight $\sum_{j \in M} v_j q_j$.

VCG Auction Protocol(Vickrey'61, Clarke'71, Groves'73)

Pricing:

- Winner y_j pays the value difference of the weight of maximum matching of $MT(M, q; N - \{y_j\}, v \backslash \{v_j\})$ and the total weight of the matching of $MT(M, q; N, v)$ with the edge (y_j, j) removed.
  - $p_j = weight[MT(M, q; N - \{y_j\}, v \backslash \{v_j\})] - (weight[MT(M, q; N, v)] - weight(y_j, j))$.
- Loser $i \notin \{y_j : j \in M\}$ pays nothing.

Compute Price

- $MT(M, q; N, v) = \{(v_1, q_1), (v_2, q_2), \cdots, (v_m, q_m)\}$
  - Its total value is $\sum_{j=1}^{m} v_j q_j$
- $MT(M, q; N - \{y_j\}, v \backslash \{v_j\}) =$
  $\{(v_1, q_1), \cdots, (v_{j-1}, q_{j-1}), (v_{j+1}, q_j), (v_{j+2}, q_{j+1}), \cdots, (v_{m+1}, q_m)\}$
  - Its total value is $\sum_{t=1}^{j-1} v_t q_t + \sum_{t=j}^{m} v_{t+1} q_t$
- $p_j = \sum_{t=j+1}^{m} v_t (q_{t-1} - q_t) + v_{m+1} q_m$
- $p_j = \sum_{t=j+1}^{m} (v_{t+1} - v_t) q_t + v_{j+1} q_j$
- Property: $v_{m+1} q_m \leq p_j \leq v_{j+1} q_j$

NOTE: intuitively, the VCG price is more than m-Vickrey and less than GSP: Prove it.

Total Revenue of the Seller

- $R = \sum_{j=1}^{m} p_j = \sum_{j=1}^{m}[\sum_{t=j+1}^{m} v_t(q_{t-1} - q_t) + v_{m+1}q_m]$.
- $= \sum_{j=1}^{m}[\sum_{t=j+1}^{m} v_t(q_{t-1} - q_t)] + mv_{m+1}q_m$.
- $= \sum_{t=2}^{m}[\sum_{j=1}^{t-1} v_t(q_{t-1} - q_t)] + mv_{m+1}q_m$.
- $= \sum_{t=2}^{m}(t-1)v_t(q_{t-1} - q_t) + mv_{m+1}q_m$.

Example 1.1

- Bidders $N = \{v_1 = 11, v_2 = 7, v_3 = 5, v_4 = 3, v_5 = 2, v_6 = 1\}$ and two items with quality $q_1 = 5, q_2 = 1$.
- $n = 6,\ m = 2.\ v_1$ wins q_1 and v_2 wins q_2.
  - $p_1 = v_2(q_1 - q_2) + v_3 * q_2 = 7 * (5 - 1) + 5 * 1 = 33$
  - $p_2 = v_3 * q_2 = 5 * 1 = 5.$
- Total revenue: $R = 33 + 5 = 38$
- Total Social Welfare: $SW = 11 * 5 + 7 * 1 = 62$

Market Equilibrium in Sponsored Search Market

The Market Equilibrium Model Again

- Individual Optimality: Every agent gets the item of its maximum utility, or has a non-positive utility on all items.
- Market clearance: All items are sold or priced at zero.

An Example for Single Item

- Bidders $N = \{11, 7, 5, 3, 2, 1\}$ and One item with quality 1.
- Equilibrium price: $p \in [7, 11]$.
- Seller would want to choose $11 - \epsilon$.
- If bidder v_1 bids $7 + \epsilon$, it wins with utility $4 - \epsilon$.
  - $p < 11$ to make sure Bidder v_1 will take it.
  - $p > 7$ to make sure Bidder v_1 wins over Bidder v_2.
- For simplicity, we may ignore ϵ, or write 7^+ and 11^-.

Example 1.1 Again

- Bidders $N = \{v_1 = 11, v_2 = 7, v_3 = 5, v_4 = 3, v_5 = 2, v_6 = 1\}$ and two items with quality $q_1 = 5, q_2 = 1$.
- VCG is an equilibrium: v_1 wins q_1 and v_2 wins q_2; $p_1 = 33$ and $p_2 = 5$.
  - Market clears:
  - Individual Optimality: Everyone has the optimal allocation under the current price.
    - $u_1(q_1) = v_1 q_1 - p_1 = 22 > u_1(q2) = v_1 q_2 - p_2 = 11 - 5 = 6.$
    - $u_2(q_2) = v_2 q_2 - p_2 = 7 * 1 - 5 = 2 \geq u_2(q_1) = 7 * 5 - 33 = 2.$
    - All other players have non-positive utilities on both items.
- Is there any other market equilibrium?

Example 1.1 Another Market Equilibrium

- Market clears: v_1 gets q_1, v_2 gets q_2.
- Prices: $p_1 = 49$ and $p_2 = 5$.
- Individual Rationality
  - $u_1(q_1) = v_1 q_1 - p_1 = 55 - 49 = 6 \geq u_1(q2) = v_1 q_2 - p_2 = 11 - 5 = 6$.
  - $u_2(q_2) = v_2 q_2 - p_2 = 7 * 1 - 5 = 2 \geq u_2(q_1) = 7 * 5 - 49 = -14$.
  - All other players have non-positive utilities on both items.

Example 1.1 Yet Another Market Equilibrium

- Market clears: v_1 gets q_1, v_2 gets q_2.
- Prices: $p_1 = 51$ and $p_2 = 7$.
- Individual Rationality
  - $u_1(q_1) = v_1 q_1 - p_1 = 55 - 51 = 4 \geq u_1(q2) = v_1 q_2 - p_2 = 11 - 7 = 4$.
  - $u_2(q_2) = v_2 q_2 - p_2 = 7 * 1 - 7 = 0 \geq u_2(q_1) = 7 * 5 - 51 = -16$.
  - All other players have non-positive utilities on both items.

Example 1.1 Many Market Equilibria

- The price vector (p_1, p_2) is a market equilibrium if and only if the following conditions hold:
  - $5 \leq p_2 \leq 7$.
  - $p_2 + 28 \leq p_1 \leq p_2 + 44$.

Proof of sufficiency (if conditions hold)

- Prove Market clearance by assigning q_1 to v_1 and q_2 to v_2.
- Prove individual Rationality:
  - For v_1 : Prove $u_1(q_1) \geq 0$ and $u_1(q_1) \geq u_1(q_2)$.
  - For v_2 : Prove $u_2(q_2) \geq 0$ and $u_2(q_2) \geq u_2(q_1)$.
  - For v_3 : Prove $u_3(q_1) \leq 0$ and $u_3(q_2) \leq 0$.
  - All other players have non-positive utilities on both items, because the claim holds for v_3.

Proof of sufficiency: Individual Rationality

- v_1 is rational:
  - $u_1(q_1) = v_1 q_1 - p_1 = 55 - p_1 \geq 0$ since $p_1 \leq 44 + p_2 \leq 51$.
  - $u_1(q_1) = v_1 q_1 - p_1 = 55 - p_1 \geq u_1(q2) = v_1 q_2 - p_2 = 11 - p_2$ since $p_1 - p_2 \leq 44$
- v_2 is rational:
  - $u_2(q_2) = v_2 q_2 - p_2 = 7 * 1 - p_2 \geq 0$ since $p_2 \leq 7$
  - $u_2(q_2) = v_2 q_2 - p_2 = 7 - q_2 \geq u_2(q_1) = 7 * 5 - p_1 = 35 - p_1$, since $p_1 - p_2 \geq 28$
- v_3 rational:
  - $u_3(q_1) = v_3 q_1 - p_1 = 25 - p_1 \leq 0$ since $p_1 - p_2 \geq 28$ & $p_2 \geq 0$.
  - $u_3(q_2) = v_3 q_2 - p_2 = 5 - p_2 \leq 0$ since $p_2 \geq 5$.

Proof of necessity: market equilibrium exists

- Prove the allocations are: q_1 to v_1 and q_2 to v_2.
- Prove individual Rationality implies the bounds
  - $5 \leq p_2 \leq 7$.
  - $p_2 + 28 \leq p_1 \leq p_2 + 44$.

Proof of necessity: market equilibrium allocation

- First v_1 and v_2 are winners by individual rationality:
  - If v_i $i \geq 3$ is a winner of q_j, $j \in \{1, 2, \}$, v_1 and v_2 will also have positive utility on q_j.
  - loser has non-positive utility by bounded rationality.
  - v_1 and v_2 must be winners
  - There are only 2 items
  - The contradiction shows v_3 cannot be a winner.
- If V_1 wins q_2 and v_2 wins q_1.
  - $v_1 q_2 - p_2 \geq v_1 q_1 - p_1$. I.e, $11 - p_2 \geq 55 - p_1$
  - $v_2 q_1 - p_1 \geq v_2 q_2 - p_2$. I.e, $35 - p_1 \geq 7 - p_2$
  - Combining above, we have $44 \leq p_1 - p_2 \leq 28$, a contradiction

Proof of necessity: Bounds on p_1 and p_2

- Known: Market equilibrium allocates q_1 to v_1 and q_2 to v_2.
- Individual Rationality: Derived the same way as in the sufficiency proof.

Equilibrium Allocation

Find an allocation (a matching) as follows:

- Sort $v_1 \geq v_2 \geq \cdots \geq v_n$
- Sort $q_1 \geq q_2 \geq \cdots \geq q_m$
- The maximum weighted matching $MT = \{(v_j, q_j) : j \in M\}$ with total weight $\sum_{j \in M} v_j q_j$.

Proof Ideas:

- v_t $(t > m)$ cannot win any item.
- $\forall i : 1 \leq i \leq m : v_i$ must win q_i, assuming that all $v_i, 1 \leq i \leq n$ $(q_j, 1 \leq j \leq m$, respectively) have different values.
- Proof is similar when there are tied values.

v_t $(t > m)$ cannot win any item.

- Proof by contradiciton.
  - If it wins, $\exists v_i (i \leq m)$ loses.
  - If it wins q_j, $0 \leq u_t(q_j) = v_t q_j - p_j$
  - Therefore, $u_i(q_j) = v_i q_j - p_j > v_t q_j - p_j = u_t(p_j) \geq 0$.
  - Individual rationality of v_i is violated.
- Therefore, winners are $v_i, 1 \leq i \leq m$.

Proof of Equilibrium Allocation: v_j wins q_j, $1 \leq j \leq m$

- Proof by backward induction.
- First prove v_m wins q_m.
  - If not, $\exists v_i$ wins q_m with $i < m$ and $\exists j < m$, v_m wins q_j.
  - Then $v_i q_m - p_m \geq v_i q_j - p_j$ and $v_m q_j - p_j \geq v_m q_m - p_m$.
  - Adding up the two inequalities, $(v_i - v_m)q_m \geq (v_i - v_m)q_j$ or $(v_i - v_m)(q_m - q_j) \geq 0$, a contradiction.
- As v_m wins q_m, we further prove v_{m-1} wins q_{m-1} in the same way.
- $\cdots$
- As v_m wins q_m, v_{m-1} wins q_{m-1}, $\cdots$, v_2 wins q_2, we have v_1 wins q_1.

Equilibrium Prices

- There are multiple equilibria in general.
- All the equilibrium price vectors form a convex set: Proof by definition.
- (Lattice Property) If both $\vec{p}$ and $\vec{p'}$ are equilibrium price vectors, so are $\min(\vec{p}, \vec{p'})$ and $\max(\vec{p}, \vec{p'})$, where min and max are taken component wise.

Lattice Property

- (Upper Semilattice Property) If both $\vec{p}$ and $\vec{p'}$ are equilibrium price vectors, so is $\max(\vec{p}, \vec{p'})$.
  - Consider p_j being the maximum of p_j, p'_j. If v_{y_j} does not covet $q_{j'}$ at price $p_{j'}$, it would not covet $q_{j'}$ at price $\max\{p_{j'}, p'_{j'}\}$.
- (Lower Semilattice Property) If both $\vec{p}$ and $\vec{p'}$ are equilibrium price vectors, so is $\min(\vec{p}, \vec{p'})$.
  - Similar but consider one's own item gets a reduced price.

Maximum Market Equilibrium

Existence

- There is a winner with utility zero at maximum revenue market equilibrium
  - If every buyer has strictly positive utility, we can increase the prices of all items by the minimum positive utility. All conditions still hold.
- There is a component wise maximum equilibrium price vector
  - Consider the maximum revenue market equilibrium price vector (because of continuity of revenue in and closed convex set property of equilibrium prices).
  - Proof by contradiction, if there is another equilibrium price vector with at least one higher priced item, the lattice property would derive another equilibrium price vector with more revenue.

Lowest Winner's Utility

- $u_m(q_m) = 0$ in the Maximum Equilibrium.
- Proof by contradiction.
  - Assume $u_m(q_m) > 0$ but $u_i(q_i) = 0$, for some $i < m$.
  - Then, $0 = u_i(q_i) \geq u_i(q_m) \geq u_m(q_m) > 0$, a contradiction.

Algorithm MAX: Computation of Maximum Equilibrium

- $p_m = v_m q_m$.
- $p_j = p_{j+1} + v_j(q_j - q_{j+1})$, for $j = 1, 2, \cdots, m-1$.
- In other word $p_j = \sum_{j'=j}^{m} v_{j'}(q_{j'} - q_{j'+1})$, where $q_{m+1} = 0$.
- Maximum Revenue $R = \sum_{j=1}^{m} \sum_{j'=j}^{m} v_{j'}(q_{j'} - q_{j'+1}) = \sum_{j'=1}^{m} \sum_{j=1}^{j'} v_{j'}(q_{j'} - q_{j'+1}) = \sum_{j'=1}^{m} j' v_{j'}(q_{j'} - q_{j'+1})$,

Correctness of Algorithm MAX

First of all, no loser will be interested in any item as the price will be more than what it can afford. There are three more conditions to check in the proof:

- Market clearance: The same maximum matching allocation
- Individual Optimality
- Maximum Revenue.

Individual Optimality

Prove by backward induction on $j : 1 \leq j \leq m$ that $u_j(q_j) \geq 0$ and v_k $(j \leq k \leq m)$ would not covet q_i $(i \neq k, j \leq i \leq m)$.

- Base case: $j = m$, p_m is correct because of the property $u_m(q_m) = 0$ for maximum equilibrium.
- Now consider $j = m - 1$.
  - By the algorithm, $v_{m-1}q_{m-1} - p_{m-1} = v_{m-1}q_m - p_m$. So v_{m-1} does not covet q_m.
  - $u_m(q_m) - u_m(q_{m-1}) = v_m(q_m - q_{m-1}) - (p_m - p_{m-1}) = (v_m - v_{m-1})(q_m - q_{m-1}) \geq 0$. So v_m does not covet q_{m-1}.
- Assume the claim is true for j, now consider $j - 1$. We need to prove that
  - v_{j-1} does not covet $q_k, j \leq k \leq m$
  - $v_k, j \leq k \leq m$ does not covet q_{j-1}

v_{j-1} does not covet $q_k, j \leq k \leq m$

- First of all $u_{j-1}(q_{j-1}) = u_{j-1}(q_j)$. It does not covet q_j.
- Now consider $k \geq j+1$.

$$\begin{aligned} u_{j-1}(q_{j-1}) - u_{j-1}(q_k) &= u_{j-1}(q_j) - u_{j-1}(q_k) \\ &= v_{j-1}(q_j - q_k) - (p_j - p_k) \\ &\geq v_j(q_j - q_k) - (p_j - p_k) \\ &= u_j(q_j) - u_j(q_k) \\ &\geq 0, \textit{by inductive hypothesis} \end{aligned}$$

$v_k : j \le k \le m$ does not covet Item q_{j-1}

- First of all, $u_j(q_j) - u_j(q_{j-1}) = v_j(q_j - q_{j-1}) - (p_j - p_{j-1}) \ge v_{j-1}(q_j - q_{j-1}) - (p_j - p_{j-1}) = 0$. v_j does not covet q_{j-1}.
- Now consider $k \ge j + 1$.

$$\begin{array}{rcl} u_k(q_{j-1}) - u_k(q_k) & = & u_k(q_{j-1}) - u_k(q_j) + u_k(q_j) - u_k(q_k) \\ & \le & u_k(q_{j-1}) - u_k(q_j), \textit{by inductive hypothesis} \\ & = & v_k(q_{j-1} - q_j) + (p_{j-1} - p_j) \\ & \le & v_{j-1}(q_{j-1} - q_j) + (p_{j-1} - p_j) \\ & = & 0 \end{array}$$

Revenue Maximality

Given any market equilibrium vector p^*, we prove the price vector p obtained in the above algorithm satisfied $p \geq p^*$. We prove $p_j^* \leq p_j$ for all $j \in M$ by backward induction.

- Base case ($p_m^* \leq p_m$): q_m is allocated to v_m. By individual optimality of market equilibrium, $u_m(p_m^*) \geq 0$. Claim holds.
- Inductive hypothesis: Assume $p_t^* \leq p_t$ for all $t : j \leq t \leq m$.
- Consider v_{j-1}.
    - By individual optimality, $u_{j-1}(q_{j-1}) \geq u_{j-1}(q_j)$ hold at price p^*. Therefore,

$$v_{j-1}q_{j-1} - p_{j-1}^* \geq v_{j-1}q_j - p_j^* \geq v_{j-1}q_j - p_j = v_{j-1}q_{j-1} - p_{j-1},$$

      where the last equation is by the algorithm MAX. $p_{j-1}^* \leq p_{j-1}$ by the first and the last term from the above inequalities.

Algorithm MIN: Computation of Minimum Equilibrium

- $p_m = v_{m+1} q_m$.
- $p_j = p_{j+1} + v_{j+1}(q_j - q_{j+1})$, for $j = 1, 2, \cdots, m-1$.
- In other word, for the minimum market equilibrium, $p_j = \sum_{j'=j}^{m} v_{j'+1}(q_{j'} - q_{j'+1})$, where $q_{m+1} = 0$.
- Minimum Market Equilibrium Revenue $R = \sum_{j=1}^{m} \sum_{j'=j}^{m} v_{j'+1}(q_{j'} - q_{j'+1}) = \sum_{j'=1}^{m} \sum_{j=1}^{j'} v_{j'+1}(q_{j'} - q_{j'+1}) = \sum_{j'=1}^{m} j' v_{j'+1}(q_{j'} - q_{j'+1})$,

Correctness of Algorithm MIN

Prove by backward induction on $j : 1 \leq j \leq m$ that $u_j(q_j) \geq 0$ and v_k $(j \leq k \leq n)$ would not covet q_i $(i \neq k, j \leq i \leq m)$.

- Base case: $j = m$, p_m is correct because none of the losers will covet any item (price higher than their utilities)
- Now consider $j = m - 1$.
  - By the algorithm, $v_m q_{m-1} - p_{m-1} = v_m q_m - p_m$. v_m does not covet q_{m-1}.
  - $u_{m-1}(q_{m-1}) - u_{m-1}(q_m) = v_{m-1}(q_{m-1} - q_m) - (p_{m-1} - p_m) = (v_{m-1} - v_m)(q_{m-1} - q_m) \geq 0$. v_{m-1} does not covet q_m.
- Assume the claim is true for j, now consider $j - 1$. We need to prove that
  - v_{j-1} does not covet $q_k, j \leq k \leq m$
  - $v_k, j \leq k \leq m$ does not covet q_{j-1}.

v_{j-1} does not covet $q_k, j \le k \le m$

- First of all $u_{j-1}(q_{j-1}) = u_{j-1}(q_j)$. It does not covet q_j.
- Now consider $k \ge j+1$.

$$
\begin{array}{rcl}
u_{j-1}(q_{j-1}) - u_{j-1}(q_k) & = & u_{j-1}(q_j) - u_{j-1}(q_k) \\
 & = & v_{j-1}(q_j - q_k) - (p_j - p_k) \\
 & \ge & v_j(q_j - q_k) - (p_j - p_k) \\
 & = & u_j(q_j) - u_j(q_k) \\
 & \ge & 0, \textit{by inductive hypothesis}
\end{array}
$$

$v_k, j \leq k \leq m$ does not covet q_{j-1}

- First of all, $u_j(q_j) - u_j(q_{j-1}) = v_j(q_j - q_{j-1}) - (p_j - p_{j-1}) \geq v_{j-1}(q_j - q_{j-1}) - (p_j - p_{j-1}) = 0$. v_j does not covet q_{j-1}.
- Now consider $k \geq j+1$.

$$
\begin{aligned}
u_k(q_{j-1}) - u_k(q_k) &= u_k(q_{j-1}) - u_k(q_j) + u_k(q_j) - u_k(q_k) \\
&\leq u_k(q_{j-1}) - u_k(q_j), \textit{by inductive hypothesis} \\
&= v_k(q_{j-1} - q_j) + (p_{j-1} - p_j) \\
&\leq v_{j-1}(q_{j-1} - q_j) + (p_{j-1} - p_j) \\
&= 0
\end{aligned}
$$

Maximum Envy-free Solution

Market Equilibrium

- Individual Optimality: Every agent gets the item of its maximum utility, or has a non-positive utility on all items.
- Market clearance: All items are sold or priced at **zero**.

Covet-Free and Envy-free Solution

- Covet-Free Solution
  - Individual Optimality: No item an agent did not gets has a larger utility than any of what it has got.
  - Market condition: All items are sold or priced at **infinity**.
- Envy-Free Solution
  - Individual envy-free: No agent would like to get what anyone else has got.
  - Market condition: All items are sold or priced at **infinity**.
- At the matching market, the two solution concepts are the same. We follow the general practice to call it the envy-free solution.

Example 1.1 Again

- Bidders $N = \{v_1 = 11, v_2 = 7, v_3 = 5, v_4 = 3, v_5 = 2, v_6 = 1\}$ and two items with quality $q_1 = 5, q_2 = 1$.
- An envy-free solution $p_2 = \infty$.
  - v_1 wins q_1, and pays $7 * 5 = 35$.
- Another envy-free solution $p_2 = \infty$.
  - v_1 wins q_1, and pays $11 * 5 = 55$.
- Yet another envy-free solution $p_1 = p_2 = \infty$.
  - Nobody wins, total revenue is zero.

Maximum Envy-Free Solution

- Individual Optimality: Every agent gets the item of its maximum utility, or has a non-positive utility on every item.
- Market condition: All items are sold or priced at **infinity**.
- Maximum total revenue

Properties of Envy-Free Solution

- If v_i, $i \geq 1$ wins, all $v_j : 1 \leq j \leq i$ wins an item.
- Proof: If $j < i$, then $v_j > v_i$ and $max_t u_j(q_t) \geq u_j(q_*) > u_i(q_*) \geq 0$. Therefore, v_j must be a winner.
- Corollary: The possible winner set of Envy-free solution are $N_i = \{v_1, v_2, \cdots, v_i\}$ for some $i : 1 \leq i \leq m$.

Ideas to Find Envy-Free Solution

- Guess Who are the winner set: one of N_i, $i = 1, 2, \cdots, m$. It is not difficult to try everyone.
- Another Idea: Knowing what are sold (the rest priced at ∞), it is a maximum market equilibrium.
- Difficulty: Which items are priced at ∞?

Dynamic Programming

- Breaking a decision down into a sequence of decision steps.
- Derive a logic/mathematical relation along the steps.
- Solve those steps to derive the final solution.

Dynamic Programming

- Let $R(i,j)$, $1 \leq i \leq n; 1 \leq j \leq m$ denote the maximum envy-free solution of the seller for buyers $\{v_1, v_2, \cdots, v_i\}$ to win items in $\{q_1, q_2, \cdots, q_j\}$.
- Consider two cases whether buyer v_i wins item q_j or not, we have the following logic relations:
- The following conditions hold:

$$R(i,j) = \max \{R(i, j-1), \quad R(i-1, j-1) + [i * v_i - (i-1) * v_{i-1}] * q_j\}$$

Correctness of Dynamic Programming Relationship

- The maximum Envy-free solution is a maximum equilibrium on the sold items $\{j_1, j_2, \cdots, j_i\}$
- Assuming i wins j_i, we have

$$
\begin{aligned}
R(i, j_i) &= \sum_{t=1}^{i-1} t v_t [q_{j_t} - q_{j_{t+1}}] + i v_i q_{j_i} \\
&= R(i-1, j_{i-1}) + q_{j_i}[i v_i - (i-1) v_{i-1}]
\end{aligned}
$$

  - The last term only depends on values we know about the winners i already and its winning item.

Base Case of Dynamic Programming Relationship

- $R(1,1) = v_1 q_1$
- $R(1,j) = \max\{R(1,j-1), v_1 * q_j\} = v_1 q_1$.
- $R(i,j) = -\infty$ for $j < i$.

MAX Envy-free Solution

- Find $(i^*, j^*) = arg\max\{R(i,j) : 1 \leq i \leq n, 1 \leq j \leq m\}$
- For items not sold in $R(i^*, j^*)$, make their prices ∞.

Maximum VCG

Optimal VCG

- It may be possible to improve the revenue of VCG when some items are removed.
- Similar to the above solution for maximum envy-free solution.
- Market condition: All items are sold or priced at **infinity**.
- Price and allocation on the sold items are according to VCG.

Properties of Optimum VCG (Minimum Market Equilibrium

- If v_i, $i \geq 1$ wins, all $v_j : 1 \leq j \leq i$ wins an item.
- Proof: Let q_* be won by v_i. If $j < i$, then $v_j > v_i$ and $max_t u_j(q_t) \geq u_j(q_*) > u_i(q_*) \geq 0$. Therefore, v_j must be a winner.
- Corollary: The possible winner set of Optimum VCG are $N_i = \{v_1, v_2, \cdots, v_i\}$ for some $i : 1 \leq i \leq m$.

Ideas to Find Optimum VCG

- Guess Who are the winner set: Let $N_i = \{1, 2, \cdots, i\}$. One of N_i, $i = 1, 2, \cdots, m$ will be the winner. It is not difficult to try everyone.
- Another Idea: Knowing what are sold (the rest priced at ∞), it is a minimum market equilibrium.
- Difficulty: Which items are priced at ∞?

Use Dynamic Programming

- Let $VCG(i,j), 1 \le i \le n; 1 \le j \le m$ denote the revenue of the Optimum VCG for every buyer in $\{v_1, v_2, \cdots, v_i\}$ to be a winner of some item in $\{q_1, q_2, \cdots, q_j\}$.
- Consider two cases whether buyer v_i wins item q_j, we have the following logic relations:
- The following conditions hold:

$$VCG(i,j) = \max \{VCG(i,j-1), \quad VCG(i-1,j-1) + [i * v_{i+1} - (i-1) * v_i] * q_j\}$$

Correctness of Dynamic Programming Relationship

- The Optimal VCG is a minimum equilibrium on the sold items $\{j_1, j_2, \cdots, j_i\}$
- Assuming i wins j_i, we have

$$\begin{aligned} VCG(i, j_i) &= \sum_{t=1}^{i-1} tv_{t+1}[q_{j_t} - q_{j_{t+1}}] + iv_{i+1}q_{j_i} \\ &= VCG(i-1, j_{i-1}) + q_{j_i}[iv_{i+1} - (i-1)v_i] \end{aligned}$$

  - The last term only depends on values we know about the winners i already and its winning item.

Base Case of Dynamic Programming Relationship

- $VCG(1,1) = v_2 q_1$.
- $\forall j \geq 2 : VCG(1,j) = v_2 q_1$.
- $\forall i \geq 2 : VCG(i,1) = -\infty$.

Example 1.1

- $N = \{11, 7, 5, 3, 2, 1\}$
- $M = \{5, 1\}$

(i,j)	j=1	j=2
i=1	35	35
i=2	$-\infty$	$\text{VCG}(1,1) + q_2(2v_3 - v_2) = 38$

Another Example

- $N = \{11, 7, 5, 3, 2, 1\}$
- $M = \{5, 3, 1\}$

(i,j)	j=1	j=2	j=3
i=1	35	35	35
i=2	$-\infty$	44	$max\{44, V(1,2) + [2v_3 - v_2]q_3\} = 44$
i=3	$-\infty$	$-\infty$	$V(2,2) + [3v_4 - 2v_3] * q_3\} = 43$

Optimum VCG

- Find $(i^*, j^*) = arg \max\{VCG(i,j) : 1 \le i \le n, 1 \le j \le m\}$
- For items not sold in $VCG(i^*, j^*)$, make their prices ∞.

Solution Concepts in GSP

Truthful Bidding May not be Nash equilibrium in GSP

- Bidders $N = \{v_1 = 11, v_2 = 7, v_3 = 5, v_4 = 3, v_5 = 2, v_6 = 1\}$ and three items with quality $q_1 = 5, q_2 = 4, q_3 = 1$.
- Truthful Bidding Outcomes
  - v_1 wins q_1, pays $7 * 5 = 35$, utility $11 * 5 - 35 = 20$
  - v_2 wins q_2, pays $5 * 4 = 20$, utility $28 - 20 = 8$
  - v_3 wins q_3, pays $3 * 1 = 3$, utility $5 * 1 - 3 = 2$.
- Deviation by v_1 to bid 6
  - v_1 wins q_2, and pays $v_3 * q_2 = 5 * 4 = 20$, improves utility to $4 * 11 - 20 = 24$.

Envious Nash Equilibrium in GSP

- Nash equilibrium bids: No bidder would change its bid to improve its utility.
- Example: $v_1 = 11, v_2 = 7, v_3 = 5, v_4 = 3,\ q_1 = 5, q_3 = 1.$
- $b_1 = 7,\ b_2 = 5,\ b_3 = 3,$
  - v_1 wins q_1, pays $5 * 5 = 25$, gets utility $11 * 5 - 25 = 30$.
  - v_2 wins q_2, pays $3 * 1 = 3$, gets utility $7 - 3 = 4$.
- It is a bidding game Nash equilibrium: No player can change its own bid to improve its own utility.
- However, v_2 would like to exchange bid with v_1.
  - It then wins q_1, pays $5 * 5 = 25$ with better utility $7 * 5 - 25 = 10$.

Nash Equilibrium in GSP with Envy-free Property

- Nash equilibrium bids: No bidder would change its bid to improve its utility.
- Example: $b_1 = 7,\ b_2 = 6,\ b_3 = 5, b_4 = 3$, where $v_1 = 11, v_2 = 7, v_3 = 5, v_4 = 3,\ q_1 = 5, q_2 = 4, q_3 = 1$.
  - v_1 wins q_1, pays $6 * 5 = 30$, gets utility $5 * 11 - 30 = 25$.
  - v_2 wins q_2, pays $5 * 4 = 20$, gets utility $7 * 4 - 20 = 8$.
  - v_3 wins q_3, pays $3 * 1$, gets utility $5 - 3 = 2$.
- No buyer can change its own bid to improve its own utility.
- No buyer can exchange bid with another to improve its own utility.

Envy-Free and Symmetric Nash Equilibrium in GSP

- Symmetric Nash Equilibrium: No buyer can exchange bid with another to improve its own utility.
- Locally Envy-free Nash Equilibrium: No buyer would like to obtain the item and pay the price of the buyer with the next item.
- The two properties are equivalent.

Symmetric Nash Equilibrium is a Market Equilibrium

- Symmetric Nash Equilibrium: No buyer can exchange bid with another to improve its own utility.
    - This is equivalent to Individual optimality. No buyer would like the item of any other, at the current price.
- All items are sold in Symmetric Nash Equilibrium

Revenue Implication

- Symmetric Nash Equilibrium has a revenue no less than VCG
- Maximum Equilibrium has a revenue no less than any symmetric Nash equilibrium.

Forward Looking Bid

- The winner of item q_i at a price p_i can win the item q_i for any bid in (p_{i-1}, p_i).
- Forward looking bid: $b' = v - \frac{q_i}{q_{i-1}}(v - b_{i+1})$.
- Idea: bid higher to guarantee at least as much as the current utility whenever a position is won with higher quality.

Properties of Forward Looking Bid

- $b_{i+1} \le b' \le v$
- If the winner of q_i wins q_t after rebid, $j < t \mapsto u'_t \ge v - p_i$.
  - $u'_t = q_t * (v - b_{t+1}) \ge q_t(v - b') \ge q_{i-1}(v - b') \ge q_i(v - b_{i+1}) = u(q_i)$

Forward Looking Nash Equilibrium in GSP

- v_i wins q_i, $i = 1, 2, \cdots, m$.
- $b_i = v_i - \frac{q_i}{q_{i-1}}(v_i - b_{i+1})$, $i = 1, 2, \cdots, m$.
- $b_{m+1} = v_{m+1}$.

Total Revenue of Forward Looking Nash Equilibrium in GSP

The forward looking Nash equilibrium has the same algorithm as the minimum market equilibrium (and VCG).

- $p_m = q_m b_{m+1} = q_m v_{m+1}$
- $p_{i-1} = b_i q_{i-1}$
- $u_i(q_i) = v_i q_i - p_i = (v_i - b_{i+1}) q_i = (v_i - b_i) q_{i-1} = u_i(q_{i-1})$.

Exercise

The Generalized Second Price Auction (GSP)

- Define the GSP protocol
  - Give q_j to the j-th highest bidder
  - and charge it the $(j+1)$-st highest bid (times the quality of the j-th highest item).
- Prove that GSP is the truthful Vickrey auction when there is only one item $m = 1$
- Prove that GSP is not truthful in general when there are more than one items $m \geq 2$
- Prove that the seller gets more in GSP than in VCG if all bid truthfully.
- Compute these values for Example 1.1.

Weighted GSP

- The utility of an advertiser depend on two parameters: its own quality, and the position the AD is placed. $\mathrm{CTR}_{ij} = a_i * b_j$.
  - AD factor a_i: representing the attractiveness of an AD, as a result of its brand or the quality of the visual effect of the AD
  - Position factor b_j: representing the prominency of the position the AD is placed.
- Given a protocol and mechanism for the simple version. How do you create one for the weighted version?

References: Click Through Rate Models

- http://wume.cse.lehigh.edu/ ovd209/wsdm/proceedings/docs/p321.
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References: GSP solution

- Benjamin Edelman, Michael Ostrovsky, and Michael Schwarz: "Internet Advertising and the Generalized Second-Price Auction: Selling Billions of Dollars Worth of Keywords". American Economic Review 97(1), 2007 pp 242-259
- H. R. Varian. Position auctions. International Journal of Industrial Organization, 2006.
- Renato Paes Leme and Eva Tardos, Pure and Bayes-Nash Price of Anarchy for Generalized Second Price Auction, 51st Annual IEEE Symposium on Foundations of Computer Science (FOCS 2010)

Today's Exercise

- Consider one digital good is up for sale. That is, one can sell as many as possible copies.
- There are n buyers, each has her or his own private value for the product (such as Windows operating system).
- The only seller will use the following protocols:
  1. The VCG protocol
  2. GSP (the i-th bidder wins the i-th best copy and pays the $(i+1)$-bidding price
  3. The maximum single price: find the maximum of $i * b_{i+1}$ where b_i is the i-th highest bid.
- Are they truthful? What is an equilibrium bid vector of the players in each of them?

Exercise

- Minimum Market Equilibrium Price:
  $p_j = \sum_{j'=j}^{m} v_{j'+1}(q_{j'} - q_{j'+1})$, where $q_{m+1} = 0$.
- Maximum Market Equilibrium Price:
  $p_j = \sum_{j'=j}^{m} v_{j'}(q_{j'} - q_{j'+1})$, where $q_{m+1} = 0$.
- Show they are truthful mechanisms or prove they are not.

Assignments

Maximum Market Equilibrium

Try one of the followings:

1. Maximum Market Equilibrium Dynamics: Starting at truthful biddings, how would the player change their bids? Assuming best response, would the process converge?
2. Prove or Disprove that the maximum revenue market equilibrium is a symmetric Nash Equilibrium in GSP.
3. What is the maximum revenue symmetric Nash equilibrium in GSP?

Maximum VCG in Sponsored Search Model

Try one of the followings:

1. How much one can gain from VCG to maximum VCG?
2. Give Examples where maximum VCG sells only one item. extend it to $k = 2, 3, \cdots, m$ items.
3. Call an case where only one item is sold VCG_1, VCG_k for k items where $k = 2, 3, \cdots, m$. How to compute them?
4. Compare maximum VCG revenue to VCG_1, VCG_2, $\cdots$, VCG_k. Derive the best bound you may obtain.
5. Design a truthful mechanism that derives a best bound in revenue against that of VCG_1. Compare your revenue against that of VCG_1. Extend to VCG_k, $k = 1, 2, \cdots, m$.