Collective Decision

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Network Flow

Network Flow Model

- A Directed Graph $G = (N, A; s, t)$.
  1. Graph with a source s and a sink t.
  2. A flow is a collection of arcs such that if it goes in a node $v \in N - \{s, t\}$, it will go out of the same v in the same number
- A flow is a maximum flow if the amount of flow going out of s is maximised.

LP Formulation of Maximum Flow

- A Directed Graph $G = (N, A; s, t)$.

$$\begin{array}{lll} \max & \sum_{(s,i)\in A} x_{s,i} & \\ s.t. & \forall i \in A - \{s,t\} & \sum_k x_{k,i} = \sum_t x_{i,t} \\ & x \leq 1 & x \geq 0 \end{array} \tag{1}$$

- A flow is a maximum flow if the amount of flow going out of s is maximised.

FORD-FULKERSON Algorithm for Network Flow

- A Directed Graph $G = (N, A; s, t)$
- initial flow $x = 0$.
- Loop
  - Construct Auxiliary Graph $H(N, A')$
    - $A' = \{e \in A : x(e) = 0, x(inverse(e)) = 1\}$, where $e' = (b, a)$ if $e = (a, b)$.
  - Find a shortest path P from s to t in H. Exit if there is no such a path.
  - revise x: $x(e) \leftarrow x(e) + 1$ if $e \in P \cap A$ and $x(e) = x(e) - 1$ if $e \in P$ and $inverse(e) \in A$.
  - End of Loop

Properties of Algorithm

1. It terminates in no more than $degree(s)$ loops.
2. It terminates with a minimum cut found.
3. The solution is a maximum solution to LP (min cut is dual, LP=DLP).
4. It extends to the non-unit capacity case.
5. Polynomial time algorithm for unit capacity networks.
6. Using LP, we can find the solution in polynomial time.
   - Perturb the objective function appropriately, the LP has a unique solution which is integer by the (exponential) network flow algorithm.

Core: A Solution Concept in Cooperative Games

An Example of Cooperative Games

- A graph of n agents $G = (V, E, w)$ with $|V| = n$, $|E| = m$ and $w : E \rightarrow N$.
- Values of Subsets $v : 2^V \rightarrow R_+$.
  - In this case, $v(S) = sum_{e \in G[S]} w(e)$

An Example of Cooperative Games

- Core of the game: $x : V \rightarrow R_+$ such that
  - $x(N) = v(N)$(=w(E)); $\forall S \subseteq V : x(S) \geq v(S)$; $x \geq 0$.
- Is there a member in Core?
  - Try $x(i) = \frac{1}{2}\sum_{i \in e \in E} w(e)$?
- How to decide whether $x : V \rightarrow Q_+$ is in Core?
- Difficulty: There are 2^n subsets of $V : |V| = n$.

Existence of the Core

- Core of the game exists iff there is no negative cut
- Decide whether $x : V \to Q_+$ is in Core can be done in polynomial time if there is no negative edge.
- Solution by maximum flow algorithm.
  - Node set of the new graph: $V' = \{s, t\} \cup V \cup E$.
  - Edge set of the new graph:
    - $\forall e = (i, j) \in E$, create $c(s, e) = v(e), c(e, i) = c(e, j) = +\infty$.
    - $\forall i \in V$, create $c(i, t) = x(i)$.
- x is in core iff solution to the network flow from s to t has capacity $v(V) = x(V)$.

Social Choice

Individual Values and Social Choice

- Underlying problem: There are a set C of candidates, and a set V of voters.
- Each voter $v \in V$ has a permutation of C written as $\prec_v$.
- We are required to find a function f that takes the preference lists of all voters, returns a choice among the candidates.
- Output: a function $f : (\pi_C)^{|V|} \rightarrow \pi_C$ as the social choice by the voters.
- Challenge: How to choose the candidate fairly?

Individual Values and Social Welfare

- Underlying problem: There are a set C of candidates, and a set V of voters.
- Each voter $v \in V$ has a permutation of C written as $\prec_v$.
- We are required to find a function f that takes the preference lists of all voters, returns a premutation of the candidates as a social preference list.
- Output: a function $f : (\pi_C)^{|V|} \rightarrow \pi_C$ as the social welfare by the voters.
- Challenge: How to define the fairness concept and find the solution?

Arrow's Properties

- Unanimity: $f(\prec, \prec, \cdots, \prec) = \prec$
- Non-dictatorship:
  - $d \in V$ is a dictator for the social welfare function f if $f(\prec_d, \cdots) = \prec_d$ no matter what are the preference lists $\prec_c$ for $c \in V - \{d\}$.
- (IIA) Independence of Irrelevant Alternatives:
  - Let V is partitioned into $V_{i,j}$ and $V_{j,i}$ such that $\forall v \in V_{i,j}$ $i <_v j$ and $\forall u \in V_{j,i}$, $i >_u j$.
  - For any set of $|V|$ preference lists, $f(\prec_v : v \in V)$ that has the same set $V_{i,j}$ and $V_{j,i}$, the outcome $\prec$ is the same on the order of $\{i, j\}$.

Arrow's Impossibility Theorem

- Any Social Welfare satisfying Unanimity and IIA is a dictatorship.
- Proof Outline
  - Pairwise Neutrality: Given $a, b, c, d \in C$, if $\forall i, j \in V$ $b \prec_i a$ iff $d \prec'_j c$, then $b \prec_{f()} a$ iff $d \prec'_{f()} c$.
  - Proof: We merge a, b, c, d $\prec_i, \prec'_i$ into one social welfare problem under $\prec''_i$ and assume $b \prec a$, and $c \neq b$.
    - Place c such that $a \prec''_i c$ and $d \prec''_i b$. At the same time, maintain the relative relationship of a, b, and c, d.
    - $a \prec'' c$ and $d \prec'' b$ by the unanimity rule.
    - By transitivity, $d \prec'' b \prec'' a \prec'' c$.
    - Since the relationship of c and d is the same in $\prec'_i$ as in $\prec''_i$ for all i, it follows that $d \prec' c$ as well.

Arrow's Impossibility Theorem–Continued

- Any Social Welfare satisfying Unanimity and IIA is a dictatorship.
- Proof Continued: Choose alternatives $a \neq b$ Consider a sequence of profiles, π^i, $i = 0, 1, 2, \cdots, n$, such that
  - for the first i players, $(j = 1, 2, \cdots, i)$, $b \prec_j^i a$
  - for the rest, $j = i + 1, i + 2, \cdots, n$, $a \prec_j^i b$.
- by Unanimity, $a \prec^0 b$ and $b \prec^n a$.
- Let $i^* = \min\{i : b \prec^i a\}$, which will be shown to be the dictator.

Arrow's Impossibility Theorem–Continued

- Let $i^* = \min\{i : b \prec^i a\}$, which will be shown to be the dictator.
- Given a profile of preferences of the players, $(\prec_i)$ and the associate social preference $\prec$, where $c \prec_{i^*} d$, we should prove that $c \prec d$.
- Let $e \notin \{c, d\}$. Create another profile $\prec'$ such that
  - if $j < i^*$
    - $e \prec'_j c \prec'_j d$ if $c \prec_j d$
    - $e \prec'_j d \prec'_j c$ if $d \prec_j c$
  - if $j = i^*$
    - $c \prec'_j e \prec'_j d$ if $c \prec_j d$
    - $d \prec'_j e \prec'_j c$ if $d \prec_j c$
  - if $i^* < j$
    - $c \prec'_j d \prec'_j e$ if $c \prec_j d$
    - $d \prec'_j c \prec'_j e$ if $d \prec_j c$

Arrow's Impossibility Theorem–Concluded

- As e and c in $\prec'$ have the same relationship as a and b in π^{i^*-1}, we have $c \prec' e$ in $\prec'$.
- Similarly, e and d in $\prec'$ have the same relationship as a and b in π^{i^*}.
- We conclude $c \prec' e \prec' d$.
- Therefore, $c \prec_{i^*} d$ implies the social decision $c \prec d$.