Stable Marriage and Linear Utility Market Equilibrium

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March 23, 2016

1. Stable Matching
2. Linear Utility Market Equilibrium

Stable Matching

Stable Matching

- Bipartite graphs of two parts $(B, G; \pi_B, \pi_G)$
- $\forall b \pi_b \in \pi_B$ is a permutation of girls in G representing the preference list of boy b.
- Similar lists for the girls $g \in G$.
- Output: a matching M such that $\forall b, M(b) \in G$
- Stability: there is no blocking pair in M.
- Blocking pair (b_1, b_2): b_i prefers $M(b_2)$ to $M(b_1)$, and $M(b_2)$ prefers b_1 to b_2.
- Solution Protocol: Boys propose algorithm(BPA).

Properties of boy propose algorithm(BPA)

- Solution is stable
- Every boy gets his most favourite girl among all stable matchings.
- Every girl gets her least favourite boy among all stable matchings.

Optimality of boy's match

Proof by contradiction,

- Let M^0 be the matching obtained by BPA.
- Let b^0 be the first boy matched to a girl g^0 in the process of BPA but rejected by his best matched girl g^1 in another stable matching M^1.
- Let b^1 be the boy held by girl g^1 at the time rejecting b^0 in the process of BPA.
- b^1 has not been rejected by his best match yet, so it prefers g^1 to anyone he is matched to in any stable matching, in particular, in M^1.
- g^1 prefers b^1 to b^0, in particular at any matching.

Optimality of boy's match–continued

- As $(b^0, g^1) \in M^1$, b^1 and g^1 are not matched in M^1.
- g^1 prefers b^1 to b^0
- b^1 prefers g^1 to anyone else he is matched to in a stable matching.
- Therefore, (b^1, g^1) is blocking in M^1.
- The claim every boy gets his best mate in a stable matching in the BPA follows by contradiction.

Exercise

- Prove BPA returns a stable matching.
- Prove every girl in BPA gets her worst mate in a stable matching.

Linear Utilities

Linear Market Agents

- # of market agents: n, $N = \{1, 2, \cdots, n\}$.
- # of goods at the market: m, $M = \{1, 2, \cdots, j, \cdots, m\}$.
- Initial endowment of agent i: $\vec{w}_i \in R^m$, $i \in N$.
  - Normalization: $\forall j \in M$, $\sum_{i \in N} w_{i,j} = 1$ $(e^T W = e^T)$
- Market allocation to agent i: $\vec{x^*}_i \in R^m$, $i \in N$.
- Linear utility function of agent i: $u_i(\vec{x}_i) = \vec{u}_i^T \vec{x}_i \in R$, $i \in N$.
- Conceptual lesson: price is for all and value is one's own.

Linear Market Equilibrium $(\vec{p}, \{\vec{x_i^*} : i \in N\})$

- Price vector $\vec{p} \in R_+^m \geq 0$
- Budget constraint for agent i: $\vec{x^*}_i^T \vec{p} \leq \vec{w}_i^T \vec{p}$
- Individual optimality:
  $\vec{x}_i^* \in \arg\max\{u_i(\vec{x}_i) : \vec{x}_i^T \vec{p} \leq \vec{w}_i^T \vec{p}, x_i \geq 0\}$
- Market clearance condition: $\sum_{i=1}^n \vec{x}_i^* \leq \sum_{i=1}^n \vec{w}_i$
- Property: If $(x_i^*)_j > 0$, then $\forall t$ & $p_t > 0 : \frac{(u_i)_j}{p_j} \geq \frac{(u_i)_t}{p_t}$

Duality of Individual Optimality

- Price vector $\vec{p} \in R_{+}^{m} \geq 0$, Budget vector of agent i: $\vec{q}_i$.
- Individual optimality:
  $\vec{x}_i^* \in \arg\max\{\vec{u}_i^T \vec{x}_i : \vec{x}_i^T \vec{p} \leq \vec{w}_i^T \vec{p}, \vec{x}_i \geq 0\}$
- $y_i^* \in \arg\min\{y_i * \vec{w}_i^T \vec{p} : y_i \vec{p} \geq \vec{u}_i, y_i \geq 0\}$
- Complementary slackness: $y_i^*(\vec{x}_i^* - \vec{w}_i)^T \vec{p} = 0$
- Market clearance condition: $\sum_{i=1}^n \vec{x}_i^* \leq \sum_{i=1}^n \vec{w}_i$.

Rewrite Linear Market Equilibrium $(\vec{p}, \{\vec{q_i^*} : i \in N\})$

- Given price vector $\vec{p} \in R_+^m \geq 0$, let q_{ij} be the amount of money spent on goods j by agent i.
- Budget constraint for agent i: $\vec{q^*}_i^T \vec{e} \leq \vec{w}_i^T \vec{p}$
- Individual optimality:
  $\vec{q}_i^* \in \arg\max\{\sum_{p_j>0} \frac{u_{ij}}{p_j} q_{ij}) : \vec{q}_i^T \vec{e} \leq \vec{w}_i^T \vec{p}, q_i \geq 0\}$
- Property: If $q_{ij}^* > 0$, then $\forall t$, $p_t > 0 : \frac{u_{ij}}{p_j} \geq \frac{u_{it}}{p_t}$
- Market clearance condition: $\sum_{i=1}^n \vec{q}_i^T \vec{e} \leq \sum_{i=1}^n \vec{p}^T \vec{w}_i$

Global Complementary Slackness

- $\vec{v} + M\vec{y} + \vec{d} * z = \vec{0}$, $\vec{v} \geq \vec{0}$, $\vec{y} \geq \vec{0}$, $z \geq 0$ and $\vec{v}^T\vec{y} = 0$.
- $\vec{v} = (\vec{s}; \vec{t}; \cdot, \vec{r}_i, \cdot)$, $\vec{y} = (\lambda; \vec{p}; \cdot, \vec{q}_i, \cdot)$.
- $\vec{d} = (-\vec{e}; \cdot, 0, \cdot)$, $\vec{b} = (-W\vec{e}; \cdot, \vec{e}, \cdot)$.
- $M\vec{y} = (\cdot, \vec{w}_i^T p - \vec{e}^T\vec{q}_i, \cdot; -\vec{p} + \sum_{i\in N}\vec{q}_i; \cdot, \lambda * u_{ij} - p_j, \cdot)$
- $\vec{0} = (\cdot, s_i - z + \vec{w}_i^T\vec{p} - \vec{e}^T\vec{q}_i, \cdot; \vec{t} - \vec{p} + \sum_{i\in N}\vec{q}_i; \cdot, r_{ij} + \lambda * u_{ij} - p_j, \cdot)$
- $\vec{q}_i^T\vec{r}_i = 0$. $\vec{p}^T\vec{t} = 0$. $\vec{s}^T\lambda = 0$.
- $0 = s_i - z + \vec{w}_i^T\vec{p} - \vec{e}^T\vec{q}_i$;
  $\vec{0} = \vec{t} - \vec{p} + \sum_{i\in N}\vec{q}_i$;
  $0 = r_{ij} + \lambda * u_{ij} - p_j$.

Examples of Market Equilibrium

- Many: $u_1(x_1) = x_{1,1}$, $u_2(x_2) = x_{2,2}$, $w_1 = (1, 0)$ and $w_2 = (1, 0)$. Any price vector is N.E.
- No equilibrium: $u_1(x_1) = x_{1,1}$ $u_2(x_2) = x_{2,1} + x_{2,2}$; $w_1 = (1, 1)$, $w_2 = (0, 1)$.
  - No matter what the price $p > 0$ is, there is no equilibrium.
  - If $p_2 = 0$, then agent 2 would want an infinite amount. Again there is no market equilibrium.

Simplification

- Everything is owned by someone: $\forall j \in M \ \exists i \in N \ w_{ij} > 0$.
- Everything is liked by someone: $\forall j \in M \exists i \in N \ u_{i,j} > 0$.
- Normalization: $\forall j \in M$, $\sum_{i \in N} w_{i,j} = 1$

Gale's Theorem

- A subset $S \subseteq N$ is self-sufficient (ss) if $\forall s \in S, u_{s,j} > 0$ implies $\forall s' \notin S : w_{s',j} = 0$. ($S$ wants nothing from $\bar{S}$.
- An ss subset S is super-self-sufficient if $\exists s \in S$ and for some $j \in M$ $w_{s,j} > 0$ but $\forall i \in S$ $u_{ij} = 0$. (Something owned by S is wanted by none in S).
- Gale Theorem: A linear economy has a competitive equilibrium if and only if no subset of agents is super-self-sufficient.

Proof of Necessity

- Let equilibrium allocation and price be x^* and price p.
- Let S be self-sufficient.
- Then all agents in S trade with each other.
- $\sum_{i \in S} \vec{x}_i^* = \sum_{i \in S} \vec{w}_i$
- Therefore, the group in S as a whole has no money to buy from outside of S.
- $p_j = 0$ for some j not wanted by anyone in S. It can be bought by someone outside unless $p_j = 0$ and utility $u_{ij} = 0$ for all $i \in \bar{S}$. As j is not wanted by anyone, it must have been eliminated by our simplification assumption.

Proof of Sufficiency

- N itself is already a ss set.
- Do we have a proper ss subset?

Competitive Equilibrium with equal income (CEEI)

- A special kind of equilibrium such that each agent has the same income
- $\forall s, t \in N,\ \vec{p} \cdot \vec{x}_s = \vec{p} \cdot \vec{x}_t$

Uniqueness of utilities in competitive equilibrium

At any equilibrium, all bundles are equivalent.

- If two equilibrium prices are the same up to scale: trivial.
- If equilibrium (p,x) and (q,y) are different,
    1. then find maximum ratio $M = q_j/p_j$.
    2. $H = \{j|q_j/p_j = M\}$.
    3. $S = \{i|y_{i,j} > 0 \textit{ for some } i \in H\}$
    4. $\vec{p} \cdot \sum_{i \in S} \vec{w}_i = \sum_{j \in H} \vec{p}^T \vec{w}_j$