Optimum Auction

Xiaotie Deng

AIMS Lab
Department of Computer Science,SJTU

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1. Preliminaries and Notations
2. Rationality
3. Feasible Mechanisms
4. Optimal Revenue Conditions
5. An Example

Preliminaries and Notations

Market Agents

- # of bidders: n, $N = \{1, 2, \cdots, n\}$
- Types (e.g., private values) of a bidder i: $t_i \in T_i = [l_i, u_i]$,
- Elements in $\times_i T_i$: type profiles of bidders
- Distribution $\mathcal{D} \in \mathcal{T} = \times_i T_i$
- $\mathcal{D}_{-i}$ for the marginal distribution of $\mathcal{D}$ over the types of all bidders, except i.
- $\mathcal{D}_{-i}(v_i)$ conditional distribution over all types of all bidders except for bidder i, conditional on bidder i's type being v_i.

Value (Type) Distribution

- The joint probability density function on T for the vector of bidder values is denoted by

$$f(\vec{t}) = \prod_{j=1}^{n} f_j(t_j) \tag{1}$$

- That for the bidder value vector other than bidder i is denoted by

$$f_{-i}(\vec{t}_{-i}) = \prod_{j \neq i} f_j(t_j) \tag{2}$$

Mechanism Design

- An allocation rule and a pricing rule based on the bid vector $\vec{b}$ of all the buyers. We also use notations,: $\forall b_i, \vec{b}_{-i}$, similar to those we did before.
- Outcome: probability of allocation for and the price charged to each bidder

Mechanism Design: Allocation $\vec{Q}$ and Pricing $\vec{p}$

- By the revelation principle, we focus on truthful biddings $\forall i : b_i(t_i) = t_i$. (Note: we will discuss the possibility bidders may deviate from this)
- $Q_i(\vec{t})$ is the allocation probability to buyer i and $p_i(\vec{t})$ is the expected payment of buyer i, taken over the randomness of the auction protocol.
- $\sum_{i=1}^{n} Q_i(\vec{t}) \leq 1, \; Q_i(\vec{t}) \geq 0.$

Reduced Form q_i of Agent Allocation Probability Q_i

- The reduced form (interim allocation) for agent winning allocation vector $\vec{q} = (q_1, q_2, \cdots, q_n)$:

$$\forall i : q_i(t_i) = \int_{\vec{T}_{-i}} Q_i(\vec{t}) f_{-i}(\vec{t}_{-i}) dt_{-i}. \tag{3}$$

- The expected payment $p_i(t_i) = \int_{\vec{T}_{-i}} p_i(\vec{t}) f_{-i}(\vec{t}_{-i}) dt_{-i}$.

The seller's revenue R

- The expected total revenue $R = \int_{\vec{T}} \sum_{i \in N} p_i(\vec{t}) f(\vec{t}) d\vec{t}$,
- Revenue maximum auction under IC and IR.
  - IC (Incentive compatible): agent utility when telling the truth is no less than that when lying.
  - IR (Individual Rationality): agent utility is no less than zero.
- Alternatively expected revenue maximized under BIC & BIR.
  - BIC (Bayesian IC): agent expected utility when telling the truth is no less than that when lying, where expectation taken over the probability space of values of other buyers.
  - BIR (Bayesian IR): agent expected utility is no less than zero, where expectation taken over the probability space of values of other buyers.

Utility of Buyers

- Determined by mechanism $(\vec{Q}, \vec{p})$, written as $U_i(Q_i, p_i, \vec{t})$.
- Interim payment $p_i(t_i) = \int_{\vec{t}_{-i} \in \vec{T}_{-i}} p_i(t_i, \vec{t}_{-i}) d\vec{t}_{-i}$.
- For each buyer i: $U_i(Q_i, p_i, \vec{t}) = Q_i(\vec{t}) \cdot t_i - p_i(\vec{t})$
- Interim utility: $U_i(q_i, p_i, t_i) = q_i(t_i) \cdot t_i - p_i(t_i)$

Rationality

Incentive compatibility (IC) Condtion

- Incentive compatibility (IC): Every player's expected utility (over the randomness of the auction protocol) is optimized by reporting his/her true value.

$$Q_i(\vec{t}) \cdot t_i - p_i(\vec{t}) \geq Q_i(s_i, \vec{t}_{-i}) \cdot t_i - p_i(s_i, \vec{t}_{-i}) \quad \forall t_i, s_i, \vec{t}_{-i}$$

- Interim Incentive compatibility (IIC): every player's expected utility (over the randomness of both the auction protocol and that of the other buyers' values) is optimized by reporting his/her true value:

$$\forall i : q_i(t_i) \cdot t_i - p_i(t_i) \geq q_i(s_i) \cdot t_i - p_i(s_i) \quad \forall t_i, s_i$$

Individual Rationality (IR)

- Individual Rationality (IR): every truth telling bidder's utility is non-negative.

$$U_i(Q_i, p_i, \vec{t}) = Q_i(\vec{t}) \cdot t_i - p_i(\vec{t}) \geq 0 \ \forall \vec{t}$$

- Interim IR (IIR): every truth telling bidder's utility is non-negative in expectation over other bidder's types.

$$q_i(t_i) \cdot t_i - p_i(t_i) \geq 0 \ \forall t_i$$

Approximate IIR and IIC

- δ-Bayesian Incentive compatibility (δ-BIC):every player's utility is near optimum by δ

$$q_i(t_i) \cdot t_i - p_i(t_i) \geq q_i(s_i) \cdot t_i - p_i(s_i) - \delta \quad \forall t_i, s_i$$

- δ-BIR: every truth telling bidder's utility is δ close to non-negative in expectation over other bidder's types.

$$q_i(t_i) \cdot t_i - p_i(t_i) + \delta \geq 0 \ \forall t_i$$

Feasible Mechanism

Feasible Mechanism

- The properties IIC and IIR together with the probability condition for allocation $\vec{Q}$ define an auction that we call feasible.

Truthful Conditions

The allocation and pricing rule $(\vec{Q}, \vec{p})$ is a feasible auction iff the following holds:

1. Monotone: $\forall i \;\; t_i \geq s_i$ implies $q_i(t_i) \geq q_i(s_i)$
2. Utility Separation: $U_i(Q_i, p_i, t_i) = U_i(Q_i, p_i, l_i) + \int_{l_i}^{t_i} q_i(s_i) ds_i$ with $U_i(Q_i, p_i, l_i) \geq 0$.
3. Probability of Winning: $\forall t \sum_{i=1}^{n} Q_i(\vec{t}) \leq 1$ and $\vec{Q}(\vec{t}) \geq 0$.

Proof of Necessity I

- Liar's interim utility: $t_i * q_i(s_i) - p_i(s_i) =$
$$s_i * q_i(s_i) - p_i(s_i) + (t_i - s_i)q_i(s_i)$$
- Applying the IIC condition, we have
$$U_i(Q_i, p_i, t_i) \geq U_i(Q_i, p_i, s_i) + (t_i - s_i)q_i(s_i).$$
- Symmetrically,
$$U_i(Q_i, p_i, s_i) \geq U_i(Q_i, p_i, t_i) + (s_i - t_i)q_i(t_i).$$
- Combining the above two inequalities, we have
$$(s_i - t_i) * (q_i(s_i) - q_i(t_i)) \geq 0$$
- Monotonicity follows: $t_i > s_i \Longrightarrow q_i(t_i) \geq q_i(s_i)$

Proof of Necessity II

- Monotone Condition implies $\forall t_i > s_i$

$$q_i(t_i) \geq \frac{U_i(Q_i, p_i, t_i) - U_i(Q_i, p_i, s_i)}{t_i - s_i} \geq q_i(s_i)$$

- It follows that $\frac{dU_i(Q_i(\vec{t}), p_i(\vec{t}), t_i)}{dt_i} = q_i(t_i)$, which is Riemann integrable by monotonicity.

$$U_i(Q_i, p_i, t_i) = U_i(Q_i, p_i, l_i) + \int_{l_i}^{t_i} q_i(s_i) ds_i \tag{4}$$

Claim 2 follows from the above and $U_i(Q_i, p_i, l_i) \geq 0$

- Therefore,
$\forall s_i \in [l_i, t_i] \quad U_i(Q_i, p_i, t_i) = U_i(Q_i, p_i, s_i) + \int_{s_i}^{t_i} q_i(s_i') ds_i'$

Proof of Sufficiency I

Sufficiency:

- IIR follows $(\forall i \forall t_i : U_i(Q_i, p_i, t_i) \geq 0)$ by the second claim.

Proof of Sufficiency II

- The interim utility by cheating is $t_i * q_i(s_i) - p_i(s_i) =$

$$s_i * q_i(s_i) - p_i(s_i) + (t_i - s_i)q_i(s_i)$$

$$= U_i(Q_i, p_i, s_i) + (t_i - s_i)q_i(s_i)$$

$$\leq U_i(Q_i, p_i, s_i) + \int_{s_i}^{t_i} q_i(s_i')ds_i' = U_i(Q_i, p_i, t_i)$$

- where the last inequality holds by the monotonicity of q_i and the utility separation property $U_i(Q_i, p_i, t_i) = U_i(Q_i, p_i, s_i) + \int_{s_i}^{t_i} q_i(s_i')ds_i'$
- IIC follows.

Proof of Sufficiency III

- Claim 3 implies the probability requirement of the allocation.

Optimal Revenue Conditions

Optimal Mechanism

- The revenue maximum auction is to maximize the seller's expected revenue R under the restriction that IIC and IIR hold.
- Equivalently, it maximizes R subject to the following conditions:
  1. Monotone: $\forall i \ \ t_i \geq s_i$ implies $q_i(t_i) \geq q_i(s_i)$
  2. Utility Separation: $U_i(Q_i, p_i, t_i) = U_i(Q_i, p_i, l_i) + \int_{l_i}^{t_i} q_i(s_i) ds_i$ with $U_i(Q_i, p_i, l_i) \geq 0$.
  3. Probability of Winning: $\forall t \sum_{i=1}^{n} Q_i(\vec{t}) \leq 1$ and $\vec{Q}(\vec{t}) \geq 0$.

Revenue

$$\begin{aligned} R &= \sum_{i \in N} \int_{\vec{T}} t_i Q_i(\vec{t}) \vec{f}(\vec{t}) d\vec{t} - \sum_{i \in N} \int_{\vec{T}} U_i(\vec{Q}, \vec{p}, \vec{t}) \vec{f}(\vec{t}) d\vec{t} \\ &= \sum_{i \in N} \int_{T_i} t_i q_i(t_i) f_i(t_i) dt_i - \sum_{i \in N} \int_{T_i} U_i(q_i, p_i, t_i) f_i(t_i) dt_i \end{aligned}$$

where

$$U_i(q_i, p_i, t_i) = U_i(q_i, p_i, l_i) + \int_{l_i}^{t_i} q_i(s_i) ds_i$$

Utility Simplified

$$\int_{l_i}^{u_i} U_i(q_i, p_i, t_i) f_i(t_i) dt_i = \int_{l_i}^{u_i} [U_i(q_i, p_i, l_i) + \int_{l_i}^{t_i} q_i(s_i) ds_i] f_i(t_i) dt_i$$

$$\begin{aligned}
\int_{\vec{T}} U_i(\vec{Q}, \vec{p}, \vec{t}) \vec{f}(\vec{t}) d\vec{t} &= U_i(q_i, p_i, l_i) + \int_{l_i}^{u_i} \int_{s_i}^{u_i} [f_i(t_i) dt_i] q_i(s_i) ds_i \\
&= U_i(q_i, p_i, l_i) + \int_{l_i}^{u_i} [1 - F_i(s_i)] q_i(s_i) ds_i \\
&= U_i(q_i, p_i, l_i) + \int_{\vec{T}} [1 - F_i(t_i)] Q_i(\vec{t}) f_{-i}(\vec{t}_{-i}) d\vec{t}
\end{aligned}$$

Revenue Simplified

$$
\begin{aligned}
R &= \sum_{i \in N} \int_{\vec{T}} t_i Q_i(\vec{t}) \vec{f}(\vec{t}) d\vec{t} - \sum_{i \in N} \int_{\vec{T}} [1 - F_i(t_i)] Q_i(\vec{t}) \vec{f}_{-i}(\vec{t}_{-i}) d\vec{t} \\
&- \sum_{i \in N} U_i(q_i, p_i, l_i) \\
&= \sum_{i \in N} \int_{\vec{T}} [t_i - \frac{1 - F_i(t_i)}{f_i(t_i)}] Q_i(\vec{t}) \vec{f}(\vec{t}) d\vec{t} - \sum_{i \in N} U_i(q_i, p_i, l_i)
\end{aligned}
$$

Maximum Revenue in Virtual Value

- Define *virtual bid* of bidder i

$$c_i(t_i) = t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \tag{5}$$

- The expected payment of buyer i in its reduced form

$$\begin{aligned} R_i = EX[P_i(\vec{Q}, \vec{p})] = EX[P_i(\vec{q})] &= \int_{\vec{T}} c_i(t_i) Q_i(\vec{t}) \vec{f}(\vec{t}) d\vec{t} \\ &= \int_{l_i}^{u_i} c_i(t_i) q_i(t_i) f_i(t_i) dt_i \end{aligned}$$

- Then the expected revenue is dependent only on allocation

$$EX[R(\vec{Q}, \vec{p})] = \sum_{i \in N} R_i = \int_{\vec{l}}^{\vec{u}} [\sum_{i \in N} c_i(t_i) Q_i(\vec{t})] \vec{f}(\vec{t}) d\vec{t} \tag{6}$$

Allocation for Optimal Revenue

- We need to find a non-decreasing solution for $\forall i : q_i(t_i)$.
- If $\forall i : c_i(t_i)$ is monotone increasing in t_i, called *regular*
- Then we set the point-wise optimal allocation rule as follows.

$$Q_i(\vec{t}) = \begin{cases} 1 & \text{if } i = i^* \equiv \arg\max\{c_j(t_j) \geq 0 : j \in N\} \\ 0 & \text{else} \end{cases} \tag{7}$$

- for which both $Q_i(t_i, \vec{t}_{-i})$ and $q_i(t_i)$ are non-decreasing in t_i.
- Threshold price: $\bar{t}_i = \arg_{t_i \in T_i} c_i(t_i) = 0$.
- Corollary: $Q_i(t_i, t_{-i}) = q_i(t_i) = 0$ for $t_i < \bar{t}_i$.

Price for Optimal Revenue

- First selecting $\vec{p}$ at the base lines $t_i = l_i$
  $p_i(l_i, t_{-i}) = l_i * Q_i(l_i, t_{-i}) = 0$
- Selecting $\vec{Q}$ such that $R(\vec{Q}, \vec{p})$ is maximized

$$
\begin{aligned}
p_i(\vec{t}) &= t_i * Q_i(\vec{t}) - \int_{l_i}^{t_i} Q_i(s_i, t_{-i}) ds_i \\
&= t_i * Q_i(\vec{t}) - \int_{\bar{t}_i}^{t_i} Q_i(s_i, t_{-i}) ds_i
\end{aligned}
$$

An Example

An Example

- Consider two buyers of value uniform in $[1,3]$ and $[2,4]$.
- Therefore, the virtual bids are $c_i(t_i) = t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$.

$$c_1(t_1) = t_1 - \frac{1 - \frac{t_1 - 1}{2}}{1/2} = t_1 - (3 - t_1) = 2t_1 - 3 \tag{8}$$

$$c_2(t_2) = t_2 - \frac{1 - \frac{t_2 - 2}{2}}{1/2} = t_2 - (4 - t_2) = 2t_2 - 4 \tag{9}$$

- $\bar{t}_1 = 1.5$ and $\bar{t}_2 = 2$.
- $Q_1(\vec{t}) = 1$ if $c_1(t_1) \geq c_2(t_2)$ and $c(t_1) > 0$ (equivalently, $t_1 + 1/2 \geq t_2 \geq 2,\ t_1 > 1.5$.)
- $Q_2(\vec{t}) = 1$ if $c(t_1) \leq c_2(t_2)$ and $c_2(t_2) > 0$, (i.e., $1 \leq t_1 \leq t_2 - 1/2$ and $t_2 > 2$).

An Example

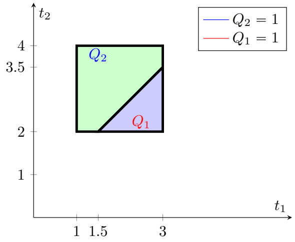
- Consider two buyers of value uniform in $[1, 3]$ and $[2, 4]$.
- Therefore, the virtual bids are $c_i(t_i) = t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$.

$$c_1(t_1) = t_1 - \frac{1 - \frac{t_1 - 1}{2}}{1/2} = t_1 - (3 - t_1) = 2t_1 - 3 \tag{10}$$

$$c_2(t_2) = t_2 - \frac{1 - \frac{t_2 - 2}{2}}{1/2} = t_2 - (4 - t_2) = 2t_2 - 4 \tag{11}$$

- $\bar{t}_1 = 1.5$ and $\bar{t}_2 = 2$.
- $Q_1(\vec{t}) = 1$ if $c_1(t_1) \geq c_2(t_2)$ and $c(t_1) > 0$ (equivalently, $t_1 + 1/2 \geq t_2 \geq 2$, $t_1 > 1.5$.)
- $Q_2(\vec{t}) = 1$ if $c(t_1) \leq c_2(t_2)$ and $c_2(t_2) > 0$, (i.e., $1 \leq t_1 \leq t_2 - 1/2$ and $t_2 > 2$).

Allocation Function



Pricing Rule

- First selecting $\vec{p}$ at the base lines $t_i = l_i$
  $p_i(l_i, t_{-i}) = l_i * Q_i(l_i, t_{-i}) = 0$
- Selecting $\vec{Q}$ such that $R(\vec{Q}, \vec{p})$ is maximized

$$p_i(\vec{t}) \quad = \quad t_i * Q_i(\vec{t}) - \int_{\bar{t}_i}^{t_i} Q_i(s_i, t_{-i}) ds_i$$

$$p_i(\vec{t}) = \begin{cases} \bar{t}_i, & \text{if } t_i \geq \bar{t}_i \geq \max_{j \neq i} t_j. \\ \max_{j \neq i} t_j, & \text{if } t_i \geq \max_{j \neq i} t_j \geq \bar{t}_i \ . \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Interim Cost by Cumulative Reduced Form

$$p_i(l_i) = l_i * q_i(l_i)$$

$$p_i(t_i) = t_i * q_i(t_i) - \int_{l_i}^{t_i} q_i(s_i) ds_i$$

Exercises

Find the reduced form and CRF function and discuss allocation rules and pricing rules for the following distributions

- Consider three buyers of value ranges in $[1, 2]$ and $[1, 2]$ in the independent uniform distribution.
- two buyers independent in $[1, 4]$ and $[2, 3]$.
- n buyers iid uniform in $[0, 1]$.
- n buyers iid in the same powerlaw distribution.
- two buyers independent in different powerlaw distributions.
- two buyers in Guassian distribution.