Game Theory and Internet Protocols

Lecture 2: Linear Programming and Zero Sum Nash Equilibrium

Xiaotie Deng

AIMS Lab
Department of Computer Science
Shanghai Jiaotong University

September 26, 2016

Duality in Linear Programming

Standard Form (P)

max $c^T x$

subject to

$$Ax \leq b,$$
$$x \geq 0.$$

A is a $m \times n$ matrix, c is a column vector of n components, b of m component.

We denote it as LP.

Dual LP (D)

LP has a corresponding dual form:

$$\min \; b^T y$$

subject to

$$y^T A \geq c^T,$$
$$y \geq 0.$$

y is a column vector of m component.

We denote it as DLP.

Weak Duality

- $LP \leq DLP$
- Proof: Let x be feasible in LP and y in DLP.
  - By prime feasibility, $Ax \leq b$.
  - Left multiplying $y \geq 0$: $y^T Ax \leq y^T b$.
  - By dual feasibility and $x \geq 0$: $y^T Ax \geq c^T x$
  - it results $c^T x \leq y^T b = b^T y$, claim holds.

Farkas Lemma

- Exactly one of the following holds:
  1. $\exists z \geq 0 : Bz = f$
  2. $\exists w : w^T B \geq 0, w^T f < 0$
- "No more than one" is easy: proof by contradiction:
  - Left multiplying both side of Item 1 by w
  - $w^T Bz = w^T f$
  - A contradiction derives as $LHS \geq 0$ and $RHS < 0$.

Prove Farkas Lemma

Proof of "either-or"

- Item 1 states f is a positive linear combination of the columns of B. That is, it is in the convex cone of the column vectors of B.
- If item 1 does not hold, there is a hyperplane $c^T z = 0$ separating f and column vectors of B.
  - $c^T f < 0$
  - $c^T B \geq 0$
- Claim holds by setting $w = c$.

Strong Duality as an LP

- If strong duality holds then there exists (x, y) such that
  - $-c^T x + b^T y \leq 0$
  - $Ax \leq b$
  - $-A^T y \leq -c$
  - $x, y \geq 0$
- Rewrite (and name it SDLP):

$$\begin{pmatrix} -c^T & +b^T \\ A & 0 \\ 0 & -A^T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 0 \\ b \\ -c \end{pmatrix}$$

$$(x, y)^T \geq 0$$

Apply Farkas Lemma

- If SDLP does not hold, by Farkas Lemma, $\exists p, q, s \geq 0$ s. t.

$$(p^T, q^T, s^T) \quad \begin{pmatrix} -c^T & +b^T \\ A & 0 \\ 0 & -A^T \end{pmatrix} \quad \geq 0$$

$$(p^T, q^T, s^T) \quad \begin{pmatrix} 0 \\ b \\ -c \end{pmatrix} \quad < 0$$

If SDLP is not True

- Simplification
  1. $-pc^T + q^T A \geq 0$
  2. $+pb^T - s^T A^T \geq 0$
  3. $q^T b - s^T c < 0$
- Right multiplying 1 by s: $-pc^T s + q^T As \geq 0$.
- Right multiplying 2 by q: $pb^T q - s^T A^T q \geq 0$.
- Add them to derive $p(b^T q - c^T s) \geq 0$.
- Combining with 3 derives $p = 0$.

If SDLP is not True

- Further Simplification
  1. $q^T A \geq 0$
  2. $As \leq 0$
  3. $q^T b - s^T c < 0$
- By 3, either $q^T b < 0$ or $c^T s > 0$.
  - If $q^T b < 0$ then a feasible *DLP* implies it is not bounded from below.
  - If $c^T s > 0$ then a feasible *LP* implies it is not bounded from above.

Full Strong Duality

- Exactly one of the following holds:
  1. $DLP = LP$
  2. LP is feasible and unbounded, DLP is not feasible
  3. DLP is feasible and unbounded, LP is not feasible
  4. Neither LP nor DLP is feasible and unbounded.
- Exercise: finish the full proof.

Algorithms

- Simplex Method by Georgia Dantzig
  - Given an initial solution, the algorithm goes from one basis solution (vertex of polytope) to another.
  - It is practical and runs fast in the average but known to take an exponential time.
- Ellipsoid Method by Leonid Khachiyan.
  - Given an initial ellipsoid containing the polytope, evaluate if the centre is in the polytope. If not, find a separating hyperplane passing through the centre. Find an ellipsoid containing an half of the original ellipsoid.
  - It is the first polynomial time algorithm for finding a solution.
- Interior Method by Narendra Karmarkar
  - Given an initial interior point in the polygon, move toward the optimum solution using a barrier function to avoid being to close to the boundary.
  - It is the first practical polynomial time algorithm for finding a

Solving Zero Sum Game by LP

Two Player Games in Strategic Form

- A Game of two players can be described by their payoff matricess
  1. Given two players: the row player and the column player
  2. Strategies of the row players are R and the strategies of the column player are C.
  3. Let the payoff matrix for the row player be A and the column player be B.
  4. if the strategy of the row player be i, the column player be j, then let $A(i,j)$ be the payoff to the row player be $A(i,j)$, and to the column player be $B(i,j)$.
- A mixed strategy $\sigma \in [0,1]^R$ satisfies $\sigma^T e = 1$ where $e = (1, 1, \cdots, 1)^T$

Max-Min Decision

- Row Player's Mixed Strategy
  1. $\max_x \min_{j=1}^n x^T A_{.j}$
  2. $e^T x = 1, \ x \geq 0.$
- Column Player's Mixed Strategy
  1. $\max_y \min_{i=1}^n B_{i.} y$
  2. $e^T y = 1, \ y \geq 0.$

LP and Dual LP

- Row Player: $\max v$
  1. $v \leq x^T A_{.j}$,
  2. $e^T x = 1,\ x \geq 0$.
- Column Player: $\min w$
  1. $w \geq A_{i.} y$
  2. $e^T y = 1,\ y \geq 0$.

Solving LP by Zero Sum Game

Recall Strong Duality as an *LP*

$$\begin{pmatrix} -c^T & +b^T \\ A & 0 \\ 0 & -A^T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 0 \\ b \\ -c \end{pmatrix}$$

$$(x, y)^T \geq 0$$

Rewrite it as

$$\begin{pmatrix} 0 & A^T & -c \\ -A & 0 & b \\ c^T & -b^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(x, y, 1)^T \geq 0$$

Solve Linear Program Using Zero Sum Game

- Multiplying z on both side and rescale:
  - $x' = x * z$, $y' = y * z$, $z' = z$ and $x' + y' + z' = 1$
  - We use the unprimed x, y, z for the primed x', y', z' for simplicity.
- We construct a payoff matrix for the row player R.
  - The payoff for the column player is its negation.

	x	y	z
x	0	A^T	-c
y	$-A$	0	b
z	c^T	$-b^T$	0

Reference:
http://www.optimization-online.org/DB_FILE/2010/06/2659.pdf

Properties of Asymmetric Zero Sum Game

- The asymmetric payoff matrix, combining with the zero sum game condition, implies that the game value is zero.
- Proof: By zero sum: $A = -B$ and asymmetry: $A = -A^T$.
- The row player's utility is $x^T Ay$
- Assume the game value $v < 0$ for the row player. Interprete v as the minimum value the row player can achieve no matter what is the strategy of the column player.
- Since the column's payoff is $x^T By = -x^T Ay = x^T A^T y = y^T Ax$, it has the same minimum value v.
- As the sum of their values must be zero, $2v = 0$ implies $v = 0$.

How to obtain solution for LP?

- Solve Zero Sum Game, we have

$$\begin{pmatrix} 0 & A^T & -c \\ -A & 0 & b \\ c^T & -b^T & 0 \end{pmatrix} \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x^* + y^* + z^* = 1, (x^*, y^*, z^*)^T \geq 0$$

- Dividing z^* on both side to obtain a solution for SDLP.

Proof of Correctness

- The payoff matrix is asymmetric. Combining it with the zero sum game condition, it implies that the game value is zero.
- It depends on $z^* \neq 0$.
- If $z^* = 0$, we have $A^T y^* \geq 0$ and $Ax^* \leq 0$, as well as $c^T x^* \geq b^T y^*$, which implies unbounded LP or unbounded DLP unless $c^T x^* = b^T y^* = 0$.
- Bonus assignment: Find out how to deal with the case $z^* = 0$.

Solve Zero Sum Game using Linear Program

- A Game of two players can be described by their payoff matrices $[A, B]$
  1. zero sum game: $A + B = 0$
  2. Row player's problem $\max\{x^T Ay : x^T e = 1, x \geq 0\}$ for fixed $y \geq 0, e^T y = 1$.
     - the opponent will minimise the above item, therefore, the row player will choose
     - $\max\{w : x^T A \geq w \cdot e, e^T x = 1, x \geq 0\}$
  3. Column player's problem $min\{x^T Ay : y \geq 0, e^T y = 1\}$ for fixed $x \geq 0, e^T x = 1$.
     - $\min\{z : Ay \geq z \cdot e, e^T y = 1, y \geq 0\}$
- Those give the primal and dual of a linear program.

Solving Linear Program

Solving Linear Program

- Give a primal linear program, how do we solve it?
  1. Simplex: going from one node to another
  2. Ellipsoid: Make a ball contain a solution, cut ball in half, cover the half ball by a ball(ellipsoid), then continue
  3. Interior point: move from one feasible solution to another.
- Polynomial time solution: ellipsoid and interior point.

Optimal solution can be reduced to feasibility

1. The optimum solution is found at a vertex of the polyhedra defined by $\{Ax \leq b, x \geq 0\}$.
2. A vertex can be written as a solution to a set of linear equations.
3. The representation of the vertex has at most the number of bits polynomial in the input data.
4. Do binary search on the optimum value of the linear program. The number iterations is bound by the number of bits of the vertex's representation.

2D Example

- For the 2D LP find a feasible solution in $Ax \leq b, x \geq 0$.
    1. Make a ball containing a point in the polyhedra
    2. Do a linear transformation to move the ball to a unit circle centered at the origin.
- Ask if the center is a solution, if yes, done.
- Cut the unit circle into two halves at the center so that the feasible region lies at one side.
- Decide which sides of the cutting line the feasible region lies.
- Create a new ellipse cover the corresponding half of the unit circle.
- Repeat.

Estimate the number of iterations

- Equation of the center $\{(x, y) : x^2 + y^2 \leq 1\}$
- Equation for ellipse covering upper half of center $\{(x, y) : \frac{x^2}{a^2} + \frac{(y-c)^2}{b^2} \leq 1\}$
- Calculate the new ellipse by setting it to touch three points on the unit circle $(-1, 0), (0, 1), (1, 0)$.
- The ratio will be $1/8^{1/4}$ which is less than 1.
- The size converges geometrically and it is close to the exact solution with high precision (in value exponentially closer) after a polynomial number of iterations.
- We then round it to the exact solution.

Summary

Solving Linear Program

- Exercise:
  - Solve the game of Rock-Paper-Sissors by LP
  - Find your favorite LP solver and demonstrate how to solve the above.
- References
  - http://www.optimization-online.org/DB_FILE/2010/06/2659.pdf