

Game Theory and Internet Protocols

Lecture 2: Linear Programming and Zero Sum Nash Equilibrium Xiaotie Deng

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Duality in Linear Programming

Standard Form (P)

$$\max c^T x$$

subject to

$$Ax \leq b,$$

$$x \geq 0.$$

A is a $m \times n$ matrix, c is a column vector of n components, b of m component.

We denote it as LP .

Dual LP (D)

LP has a corresponding dual form:

$$\min b^T y$$

subject to

$$y^T A \geq c^T,$$
$$y \geq 0.$$

y is a column vector of m component.

We denote it as DLP .

Weak Duality

- $LP \leq DLP$
- Proof: Let x be feasible in LP and y in DLP .
 - By primal feasibility, $Ax \leq b$.
 - Left multiplying $y \geq 0$: $y^T Ax \leq y^T b$.
 - By dual feasibility and $x \geq 0$: $y^T Ax \geq c^T x$
 - it results $c^T x \leq y^T b = b^T y$, claim holds.

Farkas Lemma

- Exactly one of the following holds:
 - 1 $\exists z \geq 0 : Bz = f$
 - 2 $\exists w : w^T B \geq 0, w^T f < 0$
- "No more than one" is easy: proof by contradiction:
 - Left multiplying both side of Item 1 by w
 - $w^T Bz = w^T f$
 - A contradiction derives as $LHS \geq 0$ and $RHS < 0$.

Prove Farkas Lemma

Proof of "either-or"

- Item 1 states f is a positive linear combination of the columns of B . That is, it is in the convex cone of the column vectors of B .
- If item 1 does not hold, there is a hyperplane $c^T z = 0$ separating f and column vectors of B .
 - $c^T f < 0$
 - $c^T B \geq 0$
- Claim holds by setting $w = c$.

Strong Duality as an LP

- If strong duality holds then there exists (x, y) such that
 - $-c^T x + b^T y \leq 0$
 - $Ax \leq b$
 - $-A^T y \leq -c$
 - $x, y \geq 0$
- Rewrite (and name it SDLP):

$$\begin{pmatrix} -c^T & +b^T \\ A & 0 \\ 0 & -A^T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 0 \\ b \\ -c \end{pmatrix}$$

$$(x, y)^T \geq 0$$

Apply Farkas Lemma

- If SDLP does not hold, by Farkas Lemma, $\exists p, q, s \geq 0$ s. t.

$$(p^T, q^T, s^T) \begin{pmatrix} -c^T & +b^T \\ A & 0 \\ 0 & -A^T \end{pmatrix} \geq 0$$

$$(p^T, q^T, s^T) \begin{pmatrix} 0 \\ b \\ -c \end{pmatrix} < 0$$

If SDLP is not True

- Simplification

- ① $-pc^T + q^T A \geq 0$

- ② $+pb^T - s^T A^T \geq 0$

- ③ $q^T b - s^T c < 0$

- Right multiplying 1 by s : $-pc^T s + q^T A s \geq 0$.
- Right multiplying 2 by q : $pb^T q - s^T A^T q \geq 0$.
- Add them to derive $p(b^T q - c^T s) \geq 0$.
- Combining with 3 derives $p = 0$.

If SDLP is not True

- Further Simplification
 - ① $q^T A \geq 0$
 - ② $As \leq 0$
 - ③ $q^T b - s^T c < 0$
- By 3, either $q^T b < 0$ or $c^T s > 0$.
 - If $q^T b < 0$ then a feasible *DLP* implies it is not bounded from below.
 - If $c^T s > 0$ then a feasible *LP* implies it is not bounded from above.

Full Strong Duality

- Exactly one of the following holds:
 - $DLP = LP$
 - LP is feasible and unbounded, DLP is not feasible
 - DLP is feasible and unbounded, LP is not feasible
 - Neither LP nor DLP is feasible and unbounded.
- Exercise: finish the full proof.

Algorithms

- Simplex Method by Georgia Dantzig
 - Given an initial solution, the algorithm goes from one basis solution (vertex of polytope) to another.
 - It is practical and runs fast in the average but known to take an exponential time.
- Ellipsoid Method by Leonid Khachiyan.
 - Given an initial ellipsoid containing the polytope, evaluate if the centre is in the polytope. If not, find a separating hyperplane passing through the centre. Find an ellipsoid containing an half of the original ellipsoid.
 - It is the first polynomial time algorithm for finding a solution.
- Interior Method by Narendra Karmarkar
 - Given an initial interior point in the polygon, move toward the optimum solution using a barrier function to avoid being too close to the boundary.
 - It is the first practical polynomial time algorithm for finding a

Solving Zero Sum Game by LP

Two Player Games in Strategic Form

- A Game of two players can be described by their payoff matrixes
 - 1 Given two players: the row player and the column player
 - 2 Strategies of the row players are R and the strategies of the column player are C .
 - 3 Let the payoff matrix for the row player be A and the column player be B .
 - 4 if the strategy of the row player be i , the column player be j , then let $A(i, j)$ be the payoff to the row player be $A(i, j)$, and to the column player be $B(i, j)$.
- A mixed strategy $\sigma \in [0, 1]^R$ satisfies $\sigma^T e = 1$ where $e = (1, 1, \dots, 1)^T$

Max-Min Decision

- Row Player's Mixed Strategy
 - 1 $\max_x \min_{j=1}^n x^T A_{.j}$
 - 2 $e^T x = 1, x \geq 0.$
- Column Player's Mixed Strategy
 - 1 $\max_y \min_{i=1}^n B_{i.} y$
 - 2 $e^T y = 1, y \geq 0.$

LP and Dual LP

- Row Player: $\max v$
 - 1 $v \leq x^T A_{.j}$,
 - 2 $e^T x = 1, x \geq 0$.
- Column Player: $\min w$
 - 1 $w \geq A_{i.} y$
 - 2 $e^T y = 1, y \geq 0$.

Solving LP by Zero Sum Game

Recall Strong Duality as an LP

$$\begin{pmatrix} -c^T & +b^T \\ A & 0 \\ 0 & -A^T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 0 \\ b \\ -c \end{pmatrix}$$

$$(x, y)^T \geq 0$$

Rewrite it as

$$\begin{pmatrix} 0 & A^T & -c \\ -A & 0 & b \\ c^T & -b^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(x, y, 1)^T \geq 0$$

Solve Linear Program Using Zero Sum Game

- Multiplying z on both side and rescale:
 - $x' = x * z$, $y' = y * z$, $z' = z$ and $x' + y' + z' = 1$
 - We use the unprimed x, y, z for the primed x', y', z' for simplicity.
- We construct a payoff matrix for the row player R .
 - The payoff for the column player is its negation.

	x	y	z
x	0	A^T	$-c$
y	$-A$	0	b
z	c^T	$-b^T$	0

Reference:

http://www.optimization-online.org/DB_FILE/2010/06/2659.pdf

Properties of Asymmetric Zero Sum Game

- The asymmetric payoff matrix, combining with the zero sum game condition, implies that the game value is zero.
- Proof: By zero sum: $A = -B$ and asymmetry: $A = -A^T$.
- The row player's utility is $x^T Ay$
- Assume the game value $v < 0$ for the row player. Interpret v as the minimum value the row player can achieve no matter what is the strategy of the column player.
- Since the column's payoff is $x^T By = -x^T Ay = x^T A^T y = y^T Ax$, it has the same minimum value v .
- As the sum of their values must be zero, $2v = 0$ implies $v = 0$.

How to obtain solution for LP?

- Solve Zero Sum Game, we have

$$\begin{pmatrix} 0 & A^T & -c \\ -A & 0 & b \\ c^T & -b^T & 0 \end{pmatrix} \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x^* + y^* + z^* = 1, (x^*, y^*, z^*)^T \geq 0$$

- Dividing z^* on both side to obtain a solution for SDLP.

Proof of Correctness

- The payoff matrix is asymmetric. Combining it with the zero sum game condition, it implies that the game value is zero.
- It depends on $z^* \neq 0$.
- If $z^* = 0$, we have $A^T y^* \geq 0$ and $Ax^* \leq 0$, as well as $c^T x^* \geq b^T y^*$, which implies unbounded LP or unbounded DLP unless $c^T x^* = b^T y^* = 0$.
- Bonus assignment: Find out how to deal with the case $z^* = 0$.

Solve Zero Sum Game using Linear Program

- A Game of two players can be described by their payoff matrices $[A, B]$
 - 1 zero sum game: $A + B = 0$
 - 2 Row player's problem $\max\{x^T A y : x^T e = 1, x \geq 0\}$ for fixed $y \geq 0, e^T y = 1$.
 - the opponent will minimise the above item, therefore, the row player will choose
 - $\max\{w : x^T A \geq w \cdot e, e^T x = 1, x \geq 0\}$
 - 3 Column player's problem $\min\{x^T A y : y \geq 0, e^T y = 1\}$ for fixed $x \geq 0, e^T x = 1$.
 - $\min\{z : A y \geq z \cdot e, e^T y = 1, y \geq 0\}$
- Those give the primal and dual of a linear program.

Solving Linear Program

Solving Linear Program

- Give a primal linear program, how do we solve it?
 - 1 Simplex: going from one node to another
 - 2 Ellipsoid: Make a ball contain a solution, cut ball in half, cover the half ball by a ball(ellipsoid), then continue
 - 3 Interior point: move from one feasible solution to another.
- Polynomial time solution: ellipsoid and interior point.

Optimal solution can be reduced to feasibility

- 1 The optimum solution is found at a vertex of the polyhedra defined by $\{Ax \leq b, x \geq 0\}$.
- 2 A vertex can be written as a solution to a set of linear equations.
- 3 The representation of the vertex has at most the number of bits polynomial in the input data.
- 4 Do binary search on the optimum value of the linear program. The number iterations is bound by the number of bits of the vertex's representation.

2D Example

- For the 2D LP find a feasible solution in $Ax \leq b, x \geq 0$.
 - 1 Make a ball containing a point in the polyhedra
 - 2 Do a linear transformation to move the ball to a unit circle centered at the origin.
- Ask if the center is a solution, if yes, done.
- Cut the unit circle into two halves at the center so that the feasible region lies at one side.
- Decide which sides of the cutting line the feasible region lies.
- Create a new ellipse cover the corresponding half of the unit circle.
- Repeat.

Estimate the number of iterations

- Equation of the center $\{(x, y) : x^2 + y^2 \leq 1\}$
- Equation for ellipse covering upper half of center $\{(x, y) : \frac{x^2}{a^2} + \frac{(y-c)^2}{b^2} \leq 1\}$
- Calculate the new ellipse by setting it to touch three points on the unit circle $(-1, 0), (0, 1), (1, 0)$.
- The ratio will be $1/8^{1/4}$ which is less than 1.
- The size converges geometrically and it is close to the exact solution with high precision (in value exponentially closer) after a polynomial number of iterations.
- We then round it to the exact solution.

Summary

Solving Linear Program

- Exercise:
 - Solve the game of Rock-Paper-Sissors by LP
 - Find your favorite LP solver and demonstrate how to solve the above.
- References
 - http://www.optimization-online.org/DB_FILE/2010/06/2659.pdf