Final Examination

Game Theoretical Methodology and
Technique for Internet Protocols

December 7, 2016

Problem 1

Consider the game represented in the table below, where Player 1 chooses the row and Player 2 chooses the column.

Table 1: A game of Chicken

	Swerve	Don't Swerve
Swerve	0,0	-1,1
Don't Swerve	T,-1	-2,-2

a) Find all of the pure strategy Nash equilibrium strategy profiles for this game when $T > 0$ and $T < 0$

b) If $T > 0$, there is a mixed strategy Nash equilibrium strategy profile that is not a pure strategy Nash equilibrium. Find it and find the payoffs to each player in this equilibrium.

c) In a mixed strategy Nash equilibrium with $T = 2$, which player is more likely to swerve? If $T = 2$, which player gets the higher expected payoff in equilibrium? Which player's equilibrium mixed strategy depends on T.

Problem 2

Definition:

- For a coalition game G = (N,v;x), x(C) is sum of allocation value of any player in coalition C, for a player $i \in C$,x_i is his allocation value, v(C) is the value of coalition C, $\#C$ is the number of player in coalition C.

- Dummy Player:The Coalition with him or without him has the same value.That is in a coalition game, a player i is dummy if for any coalition C ,$v(C \cap \{i\}) = v(C)$.

Seat distribution in the German Bundestag (at the beginning of each session)									
Bundestag	Session	Seats	CDU/CSU	SPD	FDP	Alliance '90 / The Greens[1]	The Left[2]	German Party	Others Sonstige
1st Bundestag	1949–1953	402	139	131	52	–	–	17	63[3]
2nd Bundestag	1953–1957	487	243	151	48	–	–	15	30[4]
3rd Bundestag	1957–1961	497	270	169	41	–	–	17	–
4th Bundestag	1961–1965	499	242	190	67	–	–	–	–
5th Bundestag	1965–1969	496	245	202	49	–	–	–	–
6th Bundestag	1969–1972	496	242	224	30	–	–	–	–
7th Bundestag	1972–1976	496	225	230	41	–	–	–	–
8th Bundestag	1976–1980	496	243	214	39	–	–	–	–
9th Bundestag	1980–1983	497	226	218	53	–	–	–	–
10th Bundestag	1983–1987	498	244	193	34	27	–	–	–
11th Bundestag	1987–1990	497	223	186	46	42	–	–	–
12th Bundestag	1990–1994	662	319	239	79	8	17	–	–
13th Bundestag	1994–1998	672	294	252	47	49	30	–	–
14th Bundestag	1998–2002	669	245	298	43	47	36	–	–
15th Bundestag	2002–2005	603	248	251	47	55	2	–	–
16th Bundestag	2005–2009	614	226	222	61	51	54	–	–
17th Bundestag	2009–2013	622	239	146	93	68	76	–	–
18th Bundestag	2013–	630	310[5]	192	–	63	64	–	–

Figure 1: Seat distribution in the German Bundestag

- Veto Player:The Coalition without him cannot win. That is,in a simple game, a player i is veto player if for any coalition C doesn't include i, $V(C) = 0$.

- monotone game:for any coalition C,D and $C \subset D, v(C) \leq v(D)$.

- simple game: for any coalition C, $v(C) = 1$ or 0.

- core: The coalition every player get his best allocation value.

- shapley value:Give a permutation of a coalition C $\pi(C)$, the set of all predecessors of i in π is called $S_\pi(i)$.Thus the shapley value of a player i is defined as:

$$\frac{1}{(\#C)!} \sum_{\pi(C)} (v(S_\pi(i) \cap \{i\}) - v(S_\pi(i)))$$

- The Banzhaf index is similar defined, for a player i, the Banzhaf index is:

$$\frac{1}{2^{\#C-1}} \sum_{D \subset (C \setminus \{i\})} (v(D \cap \{i\}) - v(D))$$

a) Prove that in a monotone cooperative game, a zero shapley player must be a dummy

b) Prove that a simple cooperative game G = (N,v) has a non-empty core if and only if it has a veto player.

c.1) Find all the veto players and dummy players in the list above and find a core situation.

c.2) Compute the Shapley value and Banzhaf Index of all parties of grand coalition in the 18th Bundestag.

Problem 3

Find the reduced form and CRF function and discuss allocation rules and pricing rules for the 2 independent bidders, one uniform in [0,3], another in [1,4].

Problem 4

Each of two players announces a nonnegative integer equal to at most 100. If $a_1 + a_2 \leq 100$, where a_i is the number announced by player i, then each player i receives payoff of a_i. If $a_1 + a_2 > 100$ and $a_i < a_j$ then player i receives ai and player j receives $100 - a_i$; if $a_1 + a_2 > 100$ and $a_i = a_j$ then each player receives 50. Show that the game is dominance solvable and find the set of surviving outcomes. (A strategic game is **dominance solvable** if all players are indifferent between all outcomes that survive the iterative procedure in which all the weakly dominated actions of each player are eliminated at each stage.)

Problem 5

Consider the case of multi-item double auction. There are n sellers with n same items and n buyers who want to buy one item. Each of the sellers and the buyers has a private value. Assume you are the auctioneer, all the sellers and buyers submit their prices simultaneously to you. You need to design a auction mechanism to determine which of the sellers and buyers can make a trade as well as how much each buyer need to pay and each seller can get.

a) Recall the four conditions in single item double auction which are Individual Rationality(IR), Budget Balancedness(BB), Truthfulness(TF) and Ex-Post Efficiency(EPE). Write down the four conditions corresponding to the multi-item double auction. You can just describe the conditions without formulation.

b) Recall Vickery's Impossibility, there are no protocol satisfies all four conditions above. Now you need to design two different mechanism which can satisfy three of the above four conditions and explain why. In other word, each of your two mechanisms can ignore one of the four conditions. **Note** that your two mechanisms should ignore **two different** conditions respectively and both of your mechanisms should **satisfy IR** (Or the mechanism will be irrational).

Problem 6

n rational buyers intend to buy m items in a VCG auction, m and n are positive integers, the value of buyer i for item j $v_{ij} = -im - j + mn + m + 1$, compute total payment.

Problem 7

Design an algorithm for finding market equilibrium of P2P network which is

- a tree (a connected graph with n vertices and $n - 1$ edges)

- a cactus (a connected graph where each edge belongs to at most one cycle)

in polynomial time of low degree. (something like $O(n^2), O(n^3)$)

Problem 8

In a digital goods auction, there are n buyers and an infinite number of copies of the same goods from a single seller. Assume the distribution of private values is uniform in [0, 100].

a) In fixed-price protocol, what is the optimal price?

b) Which one is better, fixed-price or fixed-1-winner (in another word, the 2-nd price auction), $n \geq 2$? Assume all buyers are truthful, consider seller's optimum.

c) Discuss the fixed-k-winner protocol.

d) Discuss which one is better, fixed-price or fixed-n-winner?

Problem 9

Prove that Turth-telling is not a dominant strategy under GSP.

Problem 10

Suppose the multidimensional stable marriage is defined as follows. An instance of the p-dimensional stable marriage problem (pDSM) is an ordered pair $(\mathcal{P}, L)$ with a set $\mathcal{P}$ of p disjoint parties with n elements each, and a set L of preference lists for every element (to be defined below). An element of a given party is said to be a member of that party. Let $U_{\mathcal{P}} = \bigcup_{P \in \mathcal{P}} P$ be the community of $\mathcal{P}$. For all $x \in \bigcup_{\mathcal{P}}$, let $prt(x)$ return the party of which x is a member.
Associated with each $x \in \bigcup_{\mathcal{P}}$ is a $p - 1 \times n$ strictly ordered preference array L_x of x's preferences defined as follows. For all $y \in P \in \mathcal{P}$ where $x \notin P$, let $L_x(y) = j$ denote that y is x's j^{th} preferred member of P. Let

$$
\begin{aligned}
L_x(P) &= \{y \in P \mid strictly\ ordered\ by\ L_x(y)\} \\
L_x &= \{L_x(P) \mid P \in \mathcal{P}\} \\
L &= \{L_x \mid x \in \bigcup_{\mathcal{P}}\}
\end{aligned}
$$

Furthermore, given $a, b \in P$, let $a \succ_x b$ denote that $L_x(a) < L_x(b)$ and let $a \succeq_x b$ denote that $L_x(a) \leq L_x(b)$.

A family $F \subseteq \bigcup_{\mathcal{P}}$ is a set of p elements, one member from each party. A matching $\mathcal{F}$ over $\mathcal{P}$ is a partition of $\bigcup_{\mathcal{P}}$ into n families. Elements of a single family are said to be relatives in $\mathcal{F}$. Let $rel_{\mathcal{F}}(x, P)$ return x's relative in $\mathcal{F}$ from P. For a given $\mathcal{F}$, a family $F \notin \mathcal{F}$ is blocking if and only if

1. $x \succeq_y rel_{\mathcal{F}}(y, prt(x))$, $y \succeq_x rel_{\mathcal{F}}(x, prt(y))$ for all $x, y \in F$

2. for each x, there exists $z \in F$ such that $z \succ_x rel_{\mathcal{F}}(x, prt(z))$

A matching $\mathcal{F}$ is unstable if there exists a blocking family in $\bigcup_{\mathcal{P}}$. $\mathcal{F}$ is otherwise stable.

Under the assumption above, prove that for pDSM, $(p-1)$ times of the GS algorithm yield stable matchings.

Problem 11

A zero-sum game of Blue side vs Red side is $G = \{S_1, S_2; A\}$ with Blue side's strategy set $S_1 = \{B_1, B_2, B_3\}$,

Red side's strategy set $S_2 = \{R_1, R_2, R_3\}$, tradeoff matrix $A = \begin{pmatrix} 2 & -2 & 1 \\ -1 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix}$. Is there a Pure Nash

Equilibrium for the game ? If so, please show it, Or tell why and show a Mixed Nash Equilibrium.

Problem 12

There are two nominated persons in an election and both have two possible strategies. The number of votes is dependent on their decisions. Their gain table is as follows:

	Morality	Tax-cuts
Economy	3,-3	-1,1
Society	-2,2	1,-1

Using linear programming, find the optimal strategies for both the players and prove the existence of Nash equilibrium.

Problem 13

a) Prove that for a bottleneck decomposition, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq 1$.

b) Prove that for a graph G, the maximal bottleneck pair is unique.

c) $(\mathbf{p}, \mathbf{X})$ is a market equilibrium and X is a proportion response protocol. Prove that if $x_{uv} > 0$, then we have $\frac{w_v}{p_v} = \max_{k \in \Gamma(u)} \frac{w_k}{p_k}$.

Problem 14

Consider a problem of approximate fixed point theorems. In approximate fixed point theorems, V is a real Banach space and X is a bounded and convex subset of V with non-empty interior and for F: $X \rightrightarrows X \, with \, X \subseteq V$, the set $\{x \in V \,|\, d(x, F(x)) = inf_{y \in F(x)} ||y - x|| \leq \varepsilon\}$ of $\varepsilon-$fixed points of the multifunction F on X is denoted by $FIX^\varepsilon(F)$. F is a weakly closed multifunction (that is, a multifunction closed with respect to the weak topology) such that F(x) is a non-empty and convex subset of X for each $x \in X$. Please proof that $FIX^\varepsilon(F) \neq \emptyset$, for each $\varepsilon > 0$.

Hint: Kautani-Fan-Glicksberg fixed point theorem says that there exists at least a fixed point in the topological space without linear structure

Problem 15

Suppose there are two i.i.d. buyers with identical distributions $U[0, 2]$.

a) Determine the bids and expected revenue for a 1st price auction.

b) Determine the expected revenue using the Myerson optimal auction.

Problem 16

Imagine n children playing together. The mother of these children has told them that if they get dirty there will be severe consequences. So, of course, each child wants to keep clean, but each would love to see the others get dirty. Now it happens during their play that some of the children, say k of them, get mud on their foreheads. Each can see the mud on others but not on his own forehead. So, of course, no one says a thing. Along comes the father, who says, At least m $(1 \leq m < k)$ of you has mud on your forehead, thus expressing a fact known to each of them before he spoke (if $k > 1$). The father then asks the following question, over and over: Does any of you know whether you have mud on your own forehead? Assuming that all the children are perceptive, intelligent, truthful, and that they answer simultaneously, How many questions the father ought to ask until the children with muddy foreheads all answer "Yes"? Explain why?